General Equilibrium

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From partial equilibrium to general equilibrium

- In many cases, the motivation for a policy comes from its partial equilibrium effects
- However, the general equilibrium effects, due to the interrelation between markets, may offset the effectiveness of the policy in addressing the original problem and/or cause other problems - the unintended consequences

Dutch Disease

Natural resource booms (Discovery of the large Groningen natural gas field in 1959)

$\Rightarrow$ Input demand $\uparrow$ by the mining sector

$\Rightarrow$ Wage $\uparrow$ due to larger induced demand for workers

$\Rightarrow$ Profits in manufacturing sector $\downarrow$ due to higher labour cost

Clean Energy and Wheat Price

Promotion of renewal fuels in the 2005 Energy Policy Act

$\Rightarrow$ Corn price $\uparrow$ as it is a clean bio-fuel

$\Rightarrow$ Production of wheat, soybean, etc $\downarrow$ due to the switch of farmers to corn

$\Rightarrow$ Prices for these other agriculture produce $\uparrow$ due to declining supply

Labour Market and Housing Supply Elasticity

Positive demand shock for final products

$\Rightarrow$ Demand for labour $\uparrow$ wage $\uparrow$

$\Rightarrow$ Labour supply $\uparrow$ because of immigrants

$\Rightarrow$ Demand for housing increases $\uparrow$

$\Rightarrow$ Housing price $\uparrow$

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By how much?
Elasticity of housing supply...

School Voucher

Consider the following government education program. How cheaper private school will affect housing market?

- Status quo: All household can go to public school near home. Private school has no such restriction but very expensive.
- Government program: Give voucher for private school to poor households: free money for sending kids to private schools
- What is the intended partial equilibrium effects? Private school demand? Public school demand?
- General equilibrium effect? Think about the decision of the marginal households who would be willing to live in a small apartment in a good school district if without the voucher.

General Equilibrium: History

- Walras (1874): the existence of an equilibrium price vector. number of equations = number of unknowns
- Barone (1908): But a formal argument is important if it is used for evaluating economic policy. For example, to determine the distributional effect of a change in trariffs, we should ideally be able to compare relative prices and production plans before or after the proposed change.
- Wald (1935): ass. $\mathbf{x}_1 \uparrow$, if $\mathbf{p}_2 \uparrow$, $\Rightarrow$ there is a unique solution.
- During 1940s:
  - game theory
  - the activity analysis of production, of linear programming
  - replace "differential calculus" by "convexity", from local to global

General Equilibrium: History, continued

- By late 1940s:
  - only one ingredient is lacking
  - prove the existence of the solutions to the set of inequalities: $z_l(\mathbf{p}) \leq 0$
    it is necessary to import from combinatorial topology: a series of sophisticated concepts and theorems related to fixed point of a continuous mapping.
  - Brouwer (1940): a continuous mapping from a simplex to itself: a systematic procedure for associating with each point x of the set an image x' in the same set.
  - first used by Von Neumann (1928, 1937)
- Until early 1950s: Arrow and Debreu (1954), Mckonzie (1954, 1959), Gale(1955), Nash (1950), Kuhn(1956), Nikaido(1956) –"Finding equilibrium".

Key Questions and Outline

Key questions

- How markets affect each other? (Spillover effect)
- Could all markets achieve equilibrium at the same time?
- Is the equilibrium outcome efficient?

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Outline

- Three activities in an economy: consumption, production and transaction
- Pure exchange economy
  - Consumption and transaction only
  - With or without money
  - Equilibria
- Market structure and efficiency - the Fundamental Welfare Theorems
- Introducing production
- Introducing time and uncertainty

Pure Exchange Economy: Elements

- J goods: $j = 1, 2, 3...J$
- I Consumers: $i = 1, 2, 3...I$
- Allocation/Consumption:

$$\mathbf{x}^i = (x_1^i, x_2^i, ..., x_j^i, ..., x_J^i)$$

- Endowment of two commodities

$$\mathbf{e}^i = (e_1^i, e_2^i, ..., e_j^i, ..., e_J^i)$$

Total endowment:

$$\mathbf{e} = \sum_i \mathbf{e}^i$$

- No production
- Voluntary exchange/barter

Pure Exchange Economy: Edgeworth Box

- Graphical presentation: Edgeworth box
- Two origins and four coordinates
- Feasible allocation

$$F(\mathbf{e}) \equiv \left\{ \mathbf{x} \middle| \sum_{i \in I} \mathbf{x}^i = \sum_{i \in I} \mathbf{e}^i \right\}$$

- Preferences for consumers: $\succsim_i$
- Exchange economy: $\{\succsim_i, \mathbf{e}^i\}_{i \in I}$

Pareto Efficiency

- An allocation $\mathbf{x} \in F(\mathbf{e})$ is **Pareto efficient** if there exists no other allocation $\mathbf{y} \in F(\mathbf{e})$ such that $\mathbf{y}^i \succsim \mathbf{x}^i$ for all consumers i, with at least one preference relation strict $\succ_i$.
- The process that an allocation $\mathbf{x}$ moves to $\mathbf{y}$ such that $\mathbf{y}^i \succsim \mathbf{x}^i$ for all consumers i, with at least one preference relation strict $\succ_i$ is called **Pareto improvement**.
- In other word, an allocation is Pareto efficient if there is no Pareto improvement.
- **Contract curve**: All Pareto efficient allocation in Edgeworth box. Graphically, it consists of tangency points of the two sets of ICs
  - What if the two goods are perfect complements for both consumers?
  - What if the two goods are perfect substitutes for both consumers?

Perfect Competition

Instead of barter trades and bargaining over the terms of trade, all individuals take prices as given and respond to prices only

...Imagine there exists an auctioneer or centralized market...

How would the equilibrium differ?

- Individual optimization: utility maximization with endowment (as income)
- Market clears - demand equals to supply

Formal Presentation of Competitive Equilibrium

- Consumer's problem: u^i continuous, strictly increasing and strictly quasi-concave

$$\max_{\mathbf{x}^i \in \Re^n_+} u^i(\mathbf{x}^i) \; s.t. \; \mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i$$

$\Rightarrow$ solution $\mathbf{x}^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$

- Excess demand

$$\begin{aligned} z_j(\mathbf{p}) &\equiv \sum_{i \in I} x^i_j(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - \sum_{i \in I} e^i_j \\ \mathbf{z}(\mathbf{p}) &= (z_1(\mathbf{p}), ..., z_J(\mathbf{p})) \end{aligned}$$

- Properties of $\mathbf{z}(\mathbf{p})$: Continuity, homogeneity and Walras' Law ($\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$). (Why?)

Formal Presentation of Competitive Equilibrium

- Walrasian equilibrium consists of $\mathbf{p}^* \gg 0$ such that

$$\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$$

- $\mathbf{p}^*$ is homogeneous - if $\mathbf{p}^*$ is equilibrium price vector, so it $t\mathbf{p}^*$ for $t > 0$
- Walras' Law: if all $n - 1$ markets are in equilibrium for some $\mathbf{p}$, then the nth market must be in equilibrium as well.

Example: finding Walrasian equilibrium

- Preference: $u_1(x_1^1, x_2^1) = (x_1^1)^a (x_2^1)^{1-a}$ and $u_2(x_1^2, x_2^2) = (x_1^2)^b (x_1^2)^{1-b}$
- Endowment: $\mathbf{e}^1 = (1, 0)$ and $\mathbf{e}^2 = (0, 1)$
- Walrasian equilibrium $\left(\frac{p_2}{p_1}\right)^*$?

Existence of Equilibrium

If each consumer's utility function is

- Continuous
- Strictly increasing
- Strictly quasi-concave

and that the aggregate endowment of each good $\sum_i e_j^i$ is strictly positive, then the aggregate excess demand function would satisfy

- $\mathbf{z}(\cdot)$ is continuous
- $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$
- If the prices of some but not all goods are arbitrarily close to 0, then the excess demand for at least one of those goods is arbitrarily high.

and then a Walrasian Equilibrium exists.

Existence of Walrasian Equilibrium

- Suppose $\sum_{i=1}^{I} \mathbf{e}^i \gg 0$
- If each consumer's utility function is
  - Continuous
  - Strictly increasing
  - Strictly quasi-concave

there exists $\mathbf{p}^* \gg 0$ such that there is no excess demand for any product, i.e,

$$\mathbf{z}(\mathbf{p}^*) = 0$$

Existence of Walrasian Equilibrium

- Alternatively, if each consumer's preference is
  - Continuous
  - Locally non-satiated
  - Strictly convex

  there exists $\mathbf{p}^* \geq 0$ such that there is no excess demand for any product, i.e,

$$\mathbf{z}(\mathbf{p}^*) \leq 0$$

  Since $\mathbf{p}'\mathbf{z}(\mathbf{p}) = 0$, this implies
  - $p_l^* > 0$ and $z_l(\mathbf{p}^*) = 0$ or
  - $p_l^* = 0$ and $z_l(\mathbf{p}^*) \leq 0$

Existence of WE: price normalization

- $\mathbf{z}(\mathbf{p})$ homo-0 implies that we can normalize price without affecting excess demands
- Define relative prices

$$\tilde{p}_j = \frac{p_j}{\sum_j p_j} \quad \text{and} \quad \sum_j \tilde{p}_j = 1$$

- Price vectors belong to $L-1$ dimensional unit simplex (with J vertices)

$$S^{J-1} = \{\tilde{\mathbf{p}} \in \mathcal{R}_+^J : \sum_j \tilde{p}_j = 1\}$$

- **Goal:** Find $\tilde{\mathbf{p}} \in S^{J-1}$ that would **result in zero excess demand for any product**.
- *Intuition: imagine a function of $\tilde{\mathbf{p}}$ which captures the adjustment process (upward adjustment to reduce excess demand and vice versa) and returns a new price vector as the value of the function. Then an equilibrium price would be a **fixed point** of such a function.*

Existence of WE: Brouwer fixed-point theorem (FPT)

If $f : S^{J-1} \to S^{J-1}$ is a continuous function from the unit simplex to itself, there is some X in S^{L-1} such that $X = f(X)$

Proof: Scarf (1973: the computation of Economic Equation).

- A simple case of $J = 2$, the existence of a fixed point in the unit 1-dimensional simplex S^1 of function $f : [0, 1] \to [0, 1]$
- Consider $g(x) = f(x) - x$, a fixed point is an x^* such that $g(x^*) = 0$
    - $g(0) = f(0) - 0 \geq 0$ since $f(0) \in [0, 1]$
    - $g(1) = f(1) - 1 \leq 0$ since $f(1) \in [0, 1]$
    - g is continuous $\Rightarrow \exists\ x^* \in [0, 1]$ such that $g(x^*) = 0$, and thus $f(x^*) = x^*$ (Intermediate Value Theorem)

Existence of WE: price adjustment function

Define function $\mathbf{z}^+(.)$ on S^{J-1} as

$$z_j^+(\tilde{\mathbf{p}}) = \max\{z_j(\tilde{\mathbf{p}}), 0\} \quad \textit{for} \quad j = 1, ..., J$$

Define function $f : S^{J-1} \to S^{J-1}$ as

$$f_j(\tilde{\mathbf{p}}) = \frac{\tilde{p}_j + z_j^+(\tilde{\mathbf{p}})}{\alpha(\tilde{\mathbf{p}})} \quad \text{for} \quad j = 1, ..., J$$

where

$$\alpha(\tilde{\mathbf{p}}) = \sum_j (\tilde{p}_j + z_j^+(\tilde{\mathbf{p}}))$$

Existence of WE: price adjustment function, continue

Remarks:

- $f_j(\mathbf{p}) \geq 0$ and $\sum_j f_j(\mathbf{p}) = 1$, thus $f(\tilde{\mathbf{p}}) \in S^{J-1}$
- Intuition: the relative price of a product with (higher) excess demand increases (more)

$$z_m(\tilde{\mathbf{p}}) > z_n(\tilde{\mathbf{p}}) \Rightarrow z_m^+(\tilde{\mathbf{p}}) > z_n^+(\tilde{\mathbf{p}}) \Rightarrow \frac{f_m(\tilde{\mathbf{p}})}{f_n(\tilde{\mathbf{p}})} = \frac{\tilde{p}_m + z_m^+(\tilde{\mathbf{p}})}{\tilde{p}_n + z_n^+(\tilde{\mathbf{p}})} > \frac{\tilde{p}_m}{\tilde{p}_n}$$

Existence of WE: fixed point of the f function

By Brouwers' FPT, there exists $\tilde{\mathbf{p}}^* \in S^{J-1}$ such that $\tilde{\mathbf{p}}^* = f(\tilde{\mathbf{p}}^*)$

$$\tilde{p}_j^* = \frac{\tilde{p}_j^* + z_j^+(\tilde{\mathbf{p}}^*)}{1 + \sum_j z_j^+(\tilde{\mathbf{p}}^*)}$$

Thus

$$\begin{aligned}
z_j^+(\tilde{\mathbf{p}}^*) &= \tilde{p}_j^* \sum_j z_j^+(\tilde{\mathbf{p}}^*) \\
\sum_j z_j(\tilde{\mathbf{p}}^*) z_j^+(\tilde{\mathbf{p}}^*) &= \sum_j z_j(\tilde{\mathbf{p}}^*)\tilde{p}_j^* \sum_j z_j^+(\tilde{\mathbf{p}}^*) \\
\sum_j z_j(\tilde{\mathbf{p}}^*) z_j^+(\tilde{\mathbf{p}}^*) &= 0 \quad \text{(Walras' Law)}
\end{aligned}$$

Since $z_j(\tilde{\mathbf{p}}^*) z_j^+(\tilde{\mathbf{p}}^*) \geq 0$, we must have $z_j(\tilde{\mathbf{p}}^*) \leq 0$ for $j = 1, ..., J$.
Thus all markets clear.

Existence of WE: strictly increasing utility function

So far, we have assumed the preference is locally non-satiated; thus it is possible that

- $p_j = 0$
- $z_j < 0$

If we assume each consumer's preference to be strictly increasing, then

- $p_j \to 0 \Rightarrow z_l \to \infty$

that is, the market of l won't clear if $p_l = 0$. In equilibrium, we must have $\mathbf{p}^* \gg 0$ and $\mathbf{z}(\mathbf{p}^*) = 0$

Efficiency

- Walrasian equilibrium allocation
  $\mathbf{x}(\mathbf{p}^*) = \left(\mathbf{x}^1(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^1), ..., \mathbf{x}^I(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^1)\right) \in F(\mathbf{e})$
- First welfare theorem: Walrasian Equilibrium $\rightarrow$ Pareto efficiency
  - No distortion of price
  - No externalities
  - No asymmetric information
  - No market power
- Second welfare theorem:
  Any Pareto efficient allocation can result from a Walrasian equilibrium (given that endowments can be redistributed in a lump sum way)

Barter Exchange Equilibrium

Barter Trade: Trading between consumers by negotiation.

- A trade would happen voluntarily if there exists a Pareto improvement between the consumers 1 and 2, i.e. $\exists \mathbf{y}$ such that $\sum_i \mathbf{y}^i = \sum_i \mathbf{e}^i$ and $\mathbf{y}^i \succsim \mathbf{e}^i$ for both $i = 1, 2$, with at least one consumer becomes strictly better-off.
- Blocking coalitions and core
  - "Coalition S **blocks** allocation $\mathbf{x}$" means there is an allocation $\mathbf{y}$ such that $\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$ and $\mathbf{y}^i \succsim \mathbf{x}^i$ for all $i \in S$, with at least one preference strict.
  - In words: a small group of consumers can benefit by trading among themselves (Pareto improvement).
  - An **unblocked allocation** is an allocation that there is no blocking coalition. $\Rightarrow$ An unblocked allocation is Pareto efficient
  - **Core** $C(\mathbf{e})$: the set of all unblocked feasible allocations

Core allocation and WEA

WEA - Walrasian equilibrium allocation

Question: what is the relationship between the outcome of a barter economy - allocations in the core - and the outcome of a competitive economy - the WEAs?

- In a pure exchange economy, the set of equilibrium allocations is a subset of the set of core allocations
- As the size of the economy grows, the set of core allocations will shrink and converge to the set of equilibrium allocations

Equilibrium and the core

The set of Walrasian equilibrium allocations $W(\mathbf{e}) \subset C(\mathbf{e})$

- When there are a large number of consumers, a competitive equilibrium is achieved only if no subgroup of consumers can make themselves better off by forming a coalition.
- In other words, increasing competition from a larger group of consumers reduces bargaining power by each individual.

Model setup

Elements

- There are I types of consumers - consumers of the same type have the same endowment and preference
- Start with an economy in which there is only one consumer of each type, then duplicate the economy
- An r-fold replica economy $\mathcal{E}_r$: r consumers of each of the I types, which gives a total of $r \cdot I$ consumers.
- Denote consumers of type i by iq where $q = 1, ..., r$

Edgeworth-Debreu-Scarf Limit Theorem on the Core

If $\mathbf{x} \in C_r$ for any $r = 1, 2, ...$, then $\mathbf{x}$ is a WEA for ε_1

Outline of the proof

- Equal treatment of like types in the core
  - Otherwise the "badly" treated agents can form a coalition to block the "unequal" allocation
  - Thus any allocation an r-fold replica economy is an r-fold replica of one allocation in the core of the basic economy
- Some core allocations in the basic economy can be blocked in the two-fold replica economy
  - Thus the core is shrinking as the size of the economy grows
- A WEA allocation in an r-fold replica economy $\Leftrightarrow$ a replica of a WEA in the basic economy
  - Thus the WEA prices and allocations remain "constant" as the economy grows
  - Thus the set of core allocations converge to the set of WEA allocations

Equal treatment (of like types) in the core

If $\mathbf{x}$ is an allocation in the core, then $\mathbf{x}^{iq} = \mathbf{x}^{iq'}$ for $\forall i$ and $\forall q, q' = 1, ..., r$.

- $I = 2$ and $r = 2$
- Allocation $(\mathbf{x}^{11}, \mathbf{x}^{12}, \mathbf{x}^{21}, \mathbf{x}^{22})$ is in the core, so it is feasible and $\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22} = 2\mathbf{e}^1 + 2\mathbf{e}^2$
- Suppose $\mathbf{x}^{11} \succsim^1 \mathbf{x}^{12}$ and $\mathbf{x}^{21} \succsim^2 \mathbf{x}^{22}$
- Coalition by consumer 12 and 22: allocation $\bar{\mathbf{x}}^{12} = \frac{\mathbf{x}^{11}+\mathbf{x}^{12}}{2}$ and $\bar{\mathbf{x}}^{22} = \frac{\mathbf{x}^{21}+\mathbf{x}^{22}}{2}$ is feasible and preferable.

Shrinking core - $C_1 \supseteq C_2 \supseteq C_3 \ldots \supseteq C_r \ldots$

- Starting from a basic economy, the border point in the core $\tilde{\mathbf{x}}$ will not be in the core of the two-fold replica economy ε_2
- $\tilde{\mathbf{x}}$ will be blocked by coalition of two consumers of type 1 and one consumer of type 2 in ε_2
- Construct an alternative allocation among the three to make Pareto improvement
  - Type-1: linear combination of their endowment point $\mathbf{e}^1$ and $\tilde{\mathbf{x}}^1$
  - Quasi-concave preference: type-1 consumers prefer the linear combination to $\tilde{\mathbf{x}}^1$, the core allocation in ε_1
  - Type-2: still $\tilde{\mathbf{x}}^2$
  - The alternative allocation is feasible

$$\begin{aligned} 2(\frac{1}{2}\mathbf{e}^1 + \frac{1}{2}\tilde{\mathbf{x}}^1) + \tilde{\mathbf{x}}^2 &= \mathbf{e}^1 + (\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2) \\ &= \mathbf{e}^1 + (\mathbf{e}^1 + \mathbf{e}^2) \end{aligned}$$

WEA in ε_r vs. r-fold replica of WEA in ε_1

- A WEA allocation in an r-fold replica economy $\Rightarrow$ a replica of a WEA in the basic economy
  WEA in ε_r $\Rightarrow$ it is in the core $\Rightarrow$ equal treatment and thus a replica of an allocation in the basic economy $\Rightarrow$ it is a WEA in the basic economy as MRS is the same across consumers
- A replica of a WEA in the basic economy $\Rightarrow$ WEA allocation in an r-fold replica economy
  Keep the same set of prices in the basic economy - utility maximization and market clearing conditions are satisfied in the r-fold replica economy

Adding production: a simple Robinson Crusoe economy

- One final product: banana
- One production input: labour
- Consume: banana and leisure
- Production frontier plays the role of budget line
- Where is the equilibrium?
  e.g.
$$\max_{b,\ell}\; b^{\alpha}(L-\ell)^{1-\alpha}\; s.t.\; b = A\ell^{\beta}$$

Robinson Crusoe, continue

- Now introduce an imaginary auctioneer announcing prices, and assume Mr. Crusoe the producer and Mr. Crusoe the consumer are both price takers
- Where is the competitive equilibrium?

$$\begin{aligned} producer: \max_{b,\ell} \quad & pb - w\ell \text{ s.t. } b = f(\ell) = A\ell^{\beta} \\ \Leftrightarrow \max_{\ell} \quad & pA\ell^{\beta} - w\ell \\ consumer: \max_{b,\ell} \quad & b^{\alpha}(L-\ell)^{1-\alpha} \text{ s.t. } pb = w\ell + \pi(w,p) \end{aligned}$$

$$pb = w\ell + \pi(w,p) \Leftrightarrow pb + w(L-\ell) = wL + \pi(w,p)$$

Mr. Crusoe the producer

FOC for profit maximization

$$pA\beta\ell^{\beta-1} = w$$

Solve for supply functions (labour demand and banana supply)

$$\begin{aligned}
\ell^d(w,p) &= (A\beta)^{1/(1-\beta)}\left(\frac{p}{w}\right)^{1/(1-\beta)} \\
b^s(w,p) &= (A)^{1/(1-\beta)}(\beta)^{\beta/(1-\beta)}\left(\frac{p}{w}\right)^{\beta/(1-\beta)} \\
\pi(w,p) &= A^{\frac{1}{1-\beta}}\left(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}}\right)w^{\frac{-\beta}{1-\beta}}p^{\frac{1}{1-\beta}}
\end{aligned}$$

Mr. Crusoe the consumer

- Budget $m(w,p) = wL + \pi(w,p)$
- Solution to utility maximization with Cobb-Douglas utility function

$$
\begin{aligned}
b^d(w,p) &= \frac{\alpha(wL + \pi(w,p))}{p} \\
\ell^s(w,p) &= L - \frac{(1-\alpha)(wL + \pi(w,p))}{w}
\end{aligned}
$$

Market clearing conditions

$$
\begin{aligned}
b^s(w,p) &= b^d(w,p) \\
\ell^s(w,p) &= \ell^d(w,p)
\end{aligned}
$$

Thus

$$
\begin{aligned}
(A)^{1/(1-\beta)}(\beta)^{\beta/(1-\beta)}\left(\frac{p}{w}\right)^{\beta/(1-\beta)} &= \frac{\alpha(wL+\pi(w,p))}{p} \\
(A\beta)^{1/(1-\beta)}\left(\frac{p}{w}\right)^{1/(1-\beta)} &= L-\frac{(1-\alpha)(wL+\pi(w,p))}{w}
\end{aligned}
$$

where

$$
\pi(w,p) = A^{\frac{1}{1-\beta}}\left(\beta^{\frac{\beta}{1-\beta}}-\beta^{\frac{1}{1-\beta}}\right)w^{\frac{-\beta}{1-\beta}}p^{\frac{1}{1-\beta}}
$$

$\Rightarrow \left(\frac{w}{p}\right)^* = A\beta^\beta\left(\frac{1-\alpha+\alpha\beta}{\alpha L}\right)^{1-\beta}$. Change with A? Change with L?

Comparing the social planner's solution and the WE

Both involve the tangency condition

$$\left|\frac{\Delta b}{\Delta \ell}\right| = MP_\ell^b = \left|\frac{MU_\ell}{MU_b}\right|$$

Interpretation:

- Social planner:
  Utility from the last unit of leisure consumption MU_ℓ the same as the utility from consuming the amount of banana that can be produced with the same amount of time $MP_\ell^b \cdot MU_b$
- Competitive equilibrium:
  - Producer profit maximization $MP_\ell^b = \frac{w}{p}$
  - Consumer utility maximization $\frac{MU_\ell}{MU_b} = \frac{w}{p}$

A general presentation of GE with production - firm behaviour

- There are fixed number of commodities, n
- There are fixed number of firms, J
- Firms are owned by consumers
- $\mathbf{y}^j$, an n-dimension vector, is a production plan of firm j (negative numbers represent inputs and positive numbers represent outputs)
- $\mathbf{Y}^j$ is the set of production plans of firm j
  1. $\mathbf{0} \in \mathbf{Y}^j \subseteq \Re^n$
  2. $\mathbf{Y}^j$ is closed and bounded
  3. $\mathbf{Y}^j$ is strictly convex - is CRS possible?
     - *returns to scale and cost function*

Firm behaviour, continue

Profit maximization: given $\mathbf{p} \gg \mathbf{0}$

$$\pi^j(\mathbf{p}) \equiv \max_{\mathbf{y}^j \in \mathbf{Y}^j} \mathbf{p} \cdot \mathbf{y}^j$$

- If $\mathbf{Y}^j$ satisfies property 1-3, solution $\mathbf{y}^j(p)$ exists, and is unique and continuous, so is $\pi^j(\mathbf{p})$.
- Some more properties of $\mathbf{Y}^j$
  4. No free lunch:
  $$\Re^n_+ \cap \mathbf{Y}^j = \{\mathbf{0}\}$$
  5. Free Disposal: For $\mathbf{y} \in \mathbf{Y}^j$, if $\mathbf{y}' \leq \mathbf{y}$, then $y' \in \mathbf{Y}^j$.
  6. Irreversibility: If $\mathbf{y} \in \mathbf{Y}^j$, then $-\mathbf{y} \notin \mathbf{Y}^j$.

Firm behaviour, continue

- Aggregate supply $\mathbf{Y} \equiv \left\{ \mathbf{y} | \mathbf{y} = \sum_{j \in J} \mathbf{y}^j, \ \textit{where } \mathbf{y}^j \in \mathbf{Y}^j \right\}$.
- If $\mathbf{Y}^j$ satisfies property 1-3, so does $\mathbf{Y}$. (How about property 4-6?)
- Aggregate profit maximization

$$\pi(\mathbf{p}) \equiv \max_{\bar{\mathbf{y}} \in \mathbf{Y}} \ \mathbf{p} \cdot \bar{\mathbf{y}}$$

Solution $\bar{y}$ maximizes aggregate profit $\Longleftrightarrow$ There exist $\mathbf{y}^j$, $j \in J$, such that

- $\sum_{j \in J} \mathbf{y}^j = \bar{y}$
- $\mathbf{y}^j$ maximizes the profit of firm j

Because the aggregate production set is simply the aggregation of individual firms' production sets.

Consumer behaviour

- There are I consumers
- The distribution of firm profits - $\theta^{ij} \in [0,1]$ is consumer i's proportion of profits by firm j.
  $\sum_{i \in I} \theta^{ij} = 1$
- Two sources of income: selling endowment and receiving firms' profits

$$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i + \sum_{j \in J} \theta^{ij} \pi^j(\mathbf{p}) = m^i(\mathbf{p})$$

- Utility maximization

$$max_{\mathbf{x}^i \in \Re^n_+} \; u^i(\mathbf{x}^i) \; s.t \; \mathbf{p} \cdot \mathbf{x}^i \leq m^i(\mathbf{p})$$
$$\Rightarrow \quad \mathbf{x}^i \left(\mathbf{p}, m^i(\mathbf{p})\right)$$

A unique and continuous solution exits, given continuous, strictly increasing and strictly quasi-concave preference and production sets satisfying the above assumptions.

Aggregation and Walras' Law

- The economy can be described as $(u^i, e^i, \theta^{ij}, \mathbf{Y}^j)_{i\in I, j\in J}$
- Excess demand for commodity k

$$z_k(\mathbf{p}) = \sum_{i\in I} \mathbf{x}_k^i\left(\mathbf{p}, m^i(\mathbf{p})\right) - \sum_{j\in J} \mathbf{y}_k^j(\mathbf{p}) - \sum_{i\in I} e_k^i$$

$$\mathbf{z}(\mathbf{p}) = (z_1(\mathbf{p}), ..., z_n(\mathbf{p}))$$

- Walras' Law holds, i.e, $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$

Existence of equilibrium

Given the restrictions on preference and production technology, $\mathbf{p}^* \gg \mathbf{0}$ exists such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$

- Consumption set is closed, convex and bounded below; the utility function is continuous, strictly increasing (there is no satiation point) and strictly quasi-concave. Each consumer's endowment is striclty positive.

So the solution to consumers' utility maximization problems exists and the demand function is continuous.

- The aggregate production set $\mathbf{Y}$ is closed, bounded and strictly convex; producing nothing is an option; there is free disposal.

So the solution to firms' profit maximization problems exists and the supply functions (output supply and input demand) are continuous. The equilibrium prices are nonnegative.

Welfare properties

- First welfare theorem: Walrasian equilibrium $(\mathbf{x}, \mathbf{y}, \mathbf{p}) \Rightarrow (\mathbf{x}, \mathbf{y})$ is Pareto efficient
- Second welfare theorem: For any Pareto efficient allocation $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}})$ there are income transfers $T_1, ..., T_I$ with $\sum_{i \in I} T_i = 0$, and price vector $\overline{\mathbf{p}}$ such that
  - $\widehat{\mathbf{x}^i}$ maximizes consumer i's utility, given budget $\overline{\mathbf{p}}\mathbf{x}^i \leq m^i(\overline{\mathbf{p}}) + T_i$
  - $\widehat{\mathbf{y}^j}$ maximizes firm j's profit

A GE model with production and multiple inputs

- There are J firms
- There are L inputs
- The total endowment of the economy is $\bar{\mathbf{z}} = (\bar{z}_1, ..., \bar{z}_L)$
- Firm j produces q^j with production function $f^j(\mathbf{z}^j)$ where
  - $\mathbf{z}^j = (z_1^j, ..., z_L^j)$
  - $\mathbf{f}(.)$ is concave, strictly increasing and differentiable
- Product prices are $\mathbf{p} = (p_1, ..., p_J)$
- Factor prices are denoted by $\mathbf{w} = (w_1, ..., w_L)$
- Small open economy with immobile inputs - taking output prices as given; input prices will be endogenously determined

Firm j's problem

$$max_{\mathbf{z}^j \geq 0} \; p^j f^j(\mathbf{z}^j) - \mathbf{w} \cdot \mathbf{z}^j$$

- FOC

$$p^j \frac{\partial f^j(\mathbf{z}^j)}{\partial z^j_\ell} = w_\ell \; for \; \forall j and \forall j$$

- Factor supply: $\bar{\mathbf{z}} = (\bar{z}_1, ..., \bar{z}_L)$
- Market clearing condition

$$\sum_{j \in J} \mathbf{z}^j(\mathbf{w}, \mathbf{p}) = \bar{\mathbf{z}}$$

- All together $L(J+1)$ conditions

A 2×2 example: $J = L = 2$

- Suppose the technology is of constant return to scale - $q^i = f^i(K^i, L^i)$, $i = 1, 2$, is homogeneous of degree one
- Edgeworth box with iso-quants - Pareto efficient points?
- Homothetic production function $\Rightarrow$
  - The Pareto set and the diagonal - no cuts
  - Factor intensities by the two firms along the Pareto set
  - Single intersection of a ray from the origin and the Pareto set
  - Monotone change of factor intensity and factor price ratio along the Pareto set

Firms' problem

- Production functions are homogeneous of degree one $\Rightarrow$ focus on unit input employment
- Use $c^i(r,w)$ for the unit cost and $(k^i(r,w), \ell^i(r,w))$ for the unit input combination by firm j.
- They are solutions to the following cost minimization problem

$$\begin{aligned} c^i(r,w) &\equiv \min_{k,\ell} rk + w\ell \ s.t. \ f^i(k,\ell) \geq 1 \\ &= rk^i(r,w) + w\ell^i(r,w) \end{aligned}$$

Equilibrium

- Equilibrium: w, r, q^1, q^2
- Equilibrium conditions
  - Zero profits

$$\begin{aligned} c^1(r, w) &= p^1 \\ c^2(r, w) &= p^2 \end{aligned}$$

  - Factor markets clear (full employment of factor)

$$\begin{aligned} k^1(r, w)q^1 + k^2(r, w)q^2 &= \overline{K} \\ \ell^1(r, w)q^1 + \ell^2(r, w)q^2 &= \overline{L} \end{aligned}$$

  - Four equations and four unknowns (r, w, q^1, q^2)

Finding equilibrium - factor prices

When can we solve for factor prices solely from zero profit conditions?

$$\begin{aligned} c^1(r,w) &= p^1 \\ c^2(r,w) &= p^2 \end{aligned}$$

- Unit-value curve (unit-quant curve)
- **NO** factor intensity reversals (FIR): the production of one product is **always** (for any factor prices) capital intensive than the other product
  Assuming product 2 is more capital intensive
- $\Rightarrow$ single intersection of the unit-value curves and the unit-cost curves
- We also know the equilibrium unit factor employment (k^{1*}, l^{1*}) and (k^{2*}, l^{2*}) from this practice

Factor prices - a summary

$k^{i'}$ and $l^{i'}$ represent factor requirement for producing output of one dollar in value

$$
\begin{aligned}
f^1(k^{1'}, l^{1'}) &= \frac{1}{p^1} && (1)\\
f^2(k^{2'}, l^{2'}) &= \frac{1}{p^2} && (2)\\
rk^{1'} + wl^{1'} &= 1 && (3)\\
rk^{2'} + wl^{2'} &= 1 && (4)\\
MRTS^1(k^{1'}, l^{1'}) &= \frac{w}{r} && (5)\\
MRTS^2(k^{2'}, l^{2'}) &= \frac{w}{r} && (6)
\end{aligned}
$$

$\Rightarrow k^{1'}, l^{1'}, k^{2'}, l^{2'}, r, w$ as functions of p^1 and p^2

Finding equilibrium - full employment of factors

Solve for the two unknowns from

$$\begin{aligned} p^1 k^{1'} q^1 + p^2 k^{2'} q^2 &= \overline{K} \\ p^1 l^{1'} q^1 + p^2 l^{2'} q^2 &= \overline{L} \end{aligned}$$

In vector form

$$p^1(k^{1'}, l^{1'})' q^1 + p^2(k^{2'}, l^{2'})' q^2 = (\overline{K}, \overline{L})'$$

- Cone of diversification: factor price equalization (independent of factor endowment)
- Specialization - pricing factor according to the marginal product values
- Multiple cones of diversification - factor intensity reversal

Production possibility frontier
- a special case of Leontief production function

- Suppose there are 3000 units of capital and 2000 units of labor in the economy
- The production technologies are as following

$$Q^m = min(\frac{1}{3}K^m, L^m)$$
$$Q^t = min(\frac{1}{2}K^t, \frac{1}{2}L^t)$$

- Translate into factor requirement

$$a_K^m = 3\ a_L^m = 1;\ a_K^t = 2;\ a_L^t = 2$$

- What is the implied PPF?

$$3Q^m + 2Q^t \leq 3000$$
$$Q^m + 2Q^t \leq 2000$$

- The set is convex and PPF is concave to the origin.

Production possibility frontier - constant returns to scale production function

In general, allowing factor substitution and with CRS

- Take two points within the PPF, to show that the set is convex we only need to show a combination of the two points is within the set.
- The two points represent two output profiles; behind each output profile is some specific combinations of inputs.
- Let's see what will happen if we reorganize the inputs...

PPF - the case of CRS, continue

- Let $Q^{iA} = f^i(K^{iA}, L^{iA})$ and $Q^{iB} = f^i(K^{iB}, L^{iB})$, $i = t, m$
- Make a linear combination of the factors used in producing (Q^{tA}, Q^{mA}) and (Q^{tB}, Q^{mB})
- With the linear combination of factors, one can produce $f^t(\lambda K^{tA} + (1-\lambda)K^{tB}, \lambda L^{tA} + (1-\lambda)L^{tB})$ and $f^m(\lambda K^{mA} + (1-\lambda)K^{mB}, \lambda L^{mA} + (1-\lambda)L^{mB})$
- Since the production function is concave, this is no less than $\lambda f^i(K^{iA}, L^{iA}) + (1-\lambda)f^i(K^{iB}, L^{iB})$, for $i = t, m$.
- That is, any point in between A and B is in the production possibility set.
- The set is convex and PPF is concave to origin.

Production possibility frontier

The production possibility frontier is concave to the origin

$$q^2 = g(q^1, K, L)$$

$f^i(K^i, L^i)$ being strictly increasing and concave $\Rightarrow$

$$\begin{aligned} \frac{\partial q^2}{\partial q^1} &< 0 \\ \frac{\partial^2 q^2}{\partial {q^1}^2} &< 0 \end{aligned}$$

Interpretation:
The opportunity cost of product 1 (in terms of q^2) increases with q^1; in other words, one needs to sacrifice more and more units of 2 to get one additional unit of 1.

Efficiency - revenue maximization

- In equilibrium

$$
\begin{aligned}
\frac{p^1}{p^2} &= \frac{c^1}{c^2} \\
&= \frac{MP_L^2}{MP_L^1} = \frac{MP_K^2}{MP_K^1} \\
&= \frac{MP_L^2 \Delta L + MP_K^2 \Delta K}{MP_L^1 \Delta L + MP_K^1 \Delta K} \\
&= \frac{\Delta q^2}{\Delta q^1}
\end{aligned}
$$

- This is the tangency condition between iso-revenue line and the PPF

Comparative statics 1 - change in product prices

- $p^1 \uparrow \Rightarrow$ IV1 shifts to the left
- $p^1 \uparrow \Rightarrow$ IC1 shifts to the right
- $w \uparrow$ and $r \downarrow \Rightarrow \frac{w}{r} \uparrow$
- $\frac{w}{p^1}$?

Comparative statics 1 - change in product prices

- $p^1 \uparrow \Rightarrow$ IV1 shifts to the left
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- $w \uparrow$ and $r \downarrow \Rightarrow \frac{w}{r} \uparrow$
- $\frac{w}{p^1}$?

  $\uparrow$ *as well - magnification effect*

Stopler-Samuelson Theorem

An increase in the relative price of a good will increase the real return to the factor used intensively in that good, and reduce the real return to the other factor.

$$p^i = c^i(r, w)$$

Take total derivative w.r.t. r and w

$$dp^i = k^i dr + l^i dw$$

Use $\widehat{x}$ for percentage change in variable x and θ_{fi} for the share of the cost of factor f by firm i,

$$\widehat{p}^i = \theta_{ki}\widehat{r} + \theta_{li}\widehat{w}$$

Solve for $\widehat{w}$ and $\widehat{r}$ as functions of $\widehat{p}^1$ and $\widehat{p}^2$

Stopler-Samuelson Theorem, continue

Suppose $\widehat{p}^1 > \widehat{p}^2$

$$\begin{aligned}\widehat{w} &= \frac{(\theta_{k2} - \theta_{k1})\widehat{p}^1 + \theta_{k1}(\widehat{p}^1 - \widehat{p}^2)}{\theta_{k2} - \theta_{k1}} > \widehat{p}^1 \\ \widehat{r} &= \frac{(\theta_{l2} - \theta_{l1})\widehat{p}^2 + \theta_{l2}(\widehat{p}^1 - \widehat{p}^2)}{\theta_{l2} - \theta_{l1}} < \widehat{p}^2\end{aligned}$$

- Wage increases by more than the price of the labour intensive good
- $\widehat{w} > \widehat{p}^1 > \widehat{p}^2$: real wage increases $\Rightarrow$ workers are better off
- $\widehat{p}^1 > \widehat{p}^2 > \widehat{r}$: interest rate decreases $\Rightarrow$ capital owners are worse off

Comparative statics 2 - change in factor endowment

- Change in the size of the Edgeworth box
- The new allocation point? Assuming
  - No factor intensity reversal
  - Both endowment points are in the cone of diversification
  - What is going to happen to factor prices?
- Given the factor prices, how would the factor allocation adjust?

Rybczynski Theorem

An increase in a factor endowment will increase output of the industry using it intensively and decrease the output of the other industry

$$
\begin{aligned}
k^{1*}q^1 + k^{2*}q^2 &= \overline{K} \\
l^{1*}q^1 + l^{2*}q^2 &= \overline{L}
\end{aligned}
$$

Thus

$$
\begin{aligned}
k^{1*}dq^1 + k^{2*}dq^2 &= d\overline{K} \\
l^{1*}dq^1 + l^{2*}dq^2 &= d\overline{L}
\end{aligned}
$$

Rybczynski Theorem, continue

Use $\widehat{x}$ for percentage change in variable x and use λ_{fi} for the share of factor f employed in sector i

$$\begin{aligned} \lambda_{k1}\widehat{q}^1 + \lambda_{k2}\widehat{q}^2 &= \widehat{\overline{K}} \\ \lambda_{l1}\widehat{q}^1 + \lambda_{l2}\widehat{q}^2 &= \widehat{\overline{L}} \end{aligned}$$

Solve for q^1 and q^2 as functions of $\widehat{\overline{L}}$

$$\begin{aligned} \widehat{q}^1 &= \frac{\lambda_{k2}}{\lambda_{k2} - \lambda_{l2}}\widehat{\overline{L}} > \widehat{\overline{L}} > 0 \\ \widehat{q}^2 &= \frac{-\lambda_{k1}}{\lambda_{l1} - \lambda_{k1}}\widehat{\overline{L}} < 0 \end{aligned}$$

PPF and Rybczynski line

- Change in PPF associated with an increase in labour endowment
- Rybczynski *line(?)* for labour
  - No change in factor prices
  - No change in unit factor demand
  - Capital endowment is constant

$$
\begin{aligned}
& k^{1*}q^1 + k^{2*}q^2 = \overline{K} \\
\Rightarrow \quad & k^{1*}\Delta q^1 + k^{2*}\Delta q^2 = 0 \\
\Rightarrow \quad & \frac{\Delta q^2}{\Delta q^1} = -\frac{k^{1*}}{k^{2*}}
\end{aligned}
$$

- By the same logic, Rybczynski line for capital $\frac{\Delta q^2}{\Delta q^1} = -\frac{l^{1*}}{l^{2*}}$
- Furthermore, $\frac{l^{1*}}{l^{2*}} > \frac{p^1}{p^2} > \frac{k^{1*}}{k^{2*}}$