Producer Theory

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Firm's problem

A firm – purchases inputs from input markets: cost
– produces and sells its product: revenue
– with the goal of maximizing its profit = revenue − cost

- Objective: maximizing profits
- Constraints: technology and market conditions on the input and output markets
- General rule: equalize marginal cost and marginal revenue

$$MC = MR$$

Profit maximizing rule

- Total Cost

$$TC(y) = \sum_{i=1}^{n} w_i(y)x_i^*(y)$$

$TC(y)$ is the **MINIMIZED** cost to produce y with given technology and conditions on input markets.

  - Technology: production function determines $x_i^*(y)$
  - Input markets: input prices - $w_i(y)$

- Total Revenue

$$TR(y) = p(y) \cdot y$$

$p(y)$ describes market conditions on the output market - firm's individual demand curve

  - Market demand - aggregation over consumers
  - Market structure - relationship between firms

Equalize marginal cost and marginal revenue

In general, we have

$$\begin{aligned} MC(y) &= MR(y) \\ \frac{\partial}{\partial y} TC(y) &= \frac{\partial}{\partial y} TR(y) \\ \frac{\partial}{\partial y} \left(\sum_{i=1}^{n} w_i(y) x_i^*(y) \right) &= \frac{\partial}{\partial y} (p(y)y) \end{aligned}$$

Special case - In competitive markets, prices are independent of output, i.e.

$$p(y) = p, w_i(y) = w_i$$

Production function

- Technological feasibility: production possibility set
- Production plan
    - a vector of inputs and outputs $\mathbf{y} = (y_1, .., y_m) \in \mathbf{Y}$
    - inputs $(-)$ and outputs $(+)$
- Single output production function: $f : \Re^n_+ \to \Re_+$

$$y = f(\mathbf{x}) = f(x_1, ..., x_n)$$

where $\mathbf{x} \geq 0$ and $y \geq 0$

Properties of production function

- Continuous and strictly increasing in all inputs x_i;
- $f(0) = 0$;
- Strictly quasi-concave: complementarity between inputs;
  - Strongest complementarity - Leontief production function/fixed proportion technology

    $$f(x_1, x_2) = \min(\alpha x_1, \beta x_2)$$

    - when $\alpha x_1 = \beta x_2$, one can not produce more by increasing x_1 or x_2 ALONE.
  - Weakest complementarity - linear production function/perfectly substitutable inputs

    $$f(x_1, x_2) = \alpha x_1 + \beta x_2$$

    - no matter what the combination of inputs, one can always substitute $\frac{\beta}{\alpha}$ units of x_1 for 1 unit of x_2.

Graphical presentation of production function

- Isoquant: similar to indifference curves
  - "iso" - equal; "quant" - quantity
  - take total derivative of the production function at a given output level to find the slope
- Quasi-concave ⇒ convex upper contour set
  ⇒ isoquants convex to the origin
  ⇒ diminishing slope when increasing along the x-axis
- Marginal rate of technical substitution ($MRTS$)

$$MRTS_{ij} = \frac{MP_i}{MP_j} = \frac{\partial f(\mathbf{x})/\partial x_i}{\partial f(\mathbf{x})/\partial x_j}$$

Separable Production Function

Separable production functions

- Classify inputs into a small number of groups

$$g_1 = (x_1, x_2, \ldots, x_{n_1}), g_2 = (x_{n_1+1}, \ldots, x_{n_2}), \ldots, g_m = (x_{n_{m-1}+1}, \ldots, x_n)$$

  Then allow within-group substitutability to be different from between-group substitutability.

Separable Production Function

- Weak: within group subst. indpt. of inputs in other groups, i.e. $\forall i, j \in g_s$ and $k \notin g_s$

$$\frac{\partial \left(MRTS_{ij}\right)}{\partial x_k} = \frac{\partial \left(MP_i/MP_j\right)}{\partial x_k} = 0 \text{ for}$$

- Strong: subst. between any two inputs from g_s and g_t indpt. of inputs in groups other than g_s and g_t, i.e. $\forall\ i \in g_s,\ j \in g_t$ and $k \notin g_s \cup g_t$

$$\frac{\partial \left(MRTS_{ij}\right)}{\partial x_k} = \frac{\partial \left(MP_i/MP_j\right)}{\partial x_k} = 0 \text{ for}$$

- *when $g_s = g_t$, it is the same as the case of weak separation*

CES Production Function

CES - constant elasticity of substitution

Elasticity of substitution

$$\sigma_{ij}(\mathbf{x}^0) = \frac{d\ln(x_j/x_i)}{d\ln MRTS_{ij}} = \frac{d\ln(x_j/x_i)}{d\ln(MP_i/MP_j)} = \frac{\frac{d(x_j/x_i)}{x_j/x_i}}{\frac{d(MP_i/MP_j)}{(MP_i/MP_j)}}$$

- % change in $MRTS$ vs. % change in factor ratio at $\mathbf{x}$
- σ measures how easily input factors can be substituted for one another (holding other inputs and output constant).
- Relate to the curvature of the isoquants - how fast does $MRTS$ diminish along an isoquant
  Strong(weak) substitutability: when increasing x_1 and reducing x_2 along one isoquant, $MRTS_{ij}$ - the ability of x_1 to substitute for x_2 - drops a little(a lot)
- $\sigma \in [0, +\infty)$

CES Production Function

- General form of CES

$$y = \left(\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)^{\frac{1}{\rho}}, \sum_{i=1}^{n} \alpha_i = 1$$

$$\sigma = \frac{1}{1-\rho}$$

$$\rho < 1 \quad and \quad \rho \neq 0$$

- Special cases
  - Leontief production function $\sigma \to 0$ and $\rho \to -\infty$
  - Linear production function $\sigma \to +\infty$ and $\rho \to 1$
  - Cobb-Douglas production function $\sigma \to 1$ and $\rho \to 0$

CES Production Function

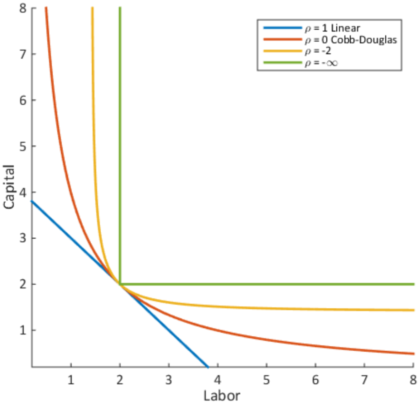

Figure: Isoquant for CES Production Functions

Returns to scale

- Returns to scale: long run concept
- Global returns to scale: for all $t > 1$ and all $\mathbf{x}$
  - Increasing: $f(t\mathbf{x}) > tf(\mathbf{x}) \Leftarrow$ homogeneous of degree α and $\alpha > 1$
  - Constant: $f(t\mathbf{x}) = tf(\mathbf{x}) \Leftrightarrow$ homogeneous of degree 1
  - Decreasing: $f(t\mathbf{x}) < tf(\mathbf{x}) \Leftarrow$ homogeneous of degree α and $\alpha < 1$
- Homogeneous production function of degree $\alpha \leq 1$ is concave
  - Homogeneous of degree 1 function is also called linear homogeneous function
  - A quasi-concave Linear homogeneous production function is concave

Linear homogeneous function is concave

Production function $y = f(\mathbf{x})$; f is quasi-concave and homogeneous of degree 1. Show that

$$f\left(t\mathbf{x} + (1-t)\mathbf{x}'\right) \geq tf(\mathbf{x}) + (1-t)f(\mathbf{x}') = ty + (1-t)y'$$

Proof:

Linear homogeneity $\Rightarrow$

$$f\left(\frac{t\mathbf{x}}{ty}\right) = f\left(\frac{(1-t)\mathbf{x}'}{(1-t)y'}\right) = 1$$

Quasi-concave $\Rightarrow$

$$f\left(\lambda\frac{t\mathbf{x}}{ty} + (1-\lambda)\frac{(1-t)\mathbf{x}'}{(1-t)y'}\right) \geq 1 \; for \; \forall\lambda \in [0,1]$$

Let $\lambda = \frac{ty}{ty+(1-t)y'}$ $\Rightarrow$

$$f\left(t\mathbf{x} + (1-t)\mathbf{x}'\right) \geq ty + (1-t)y'$$

Returns to scale, continue

Local returns to scale - elasticity of scale at $\mathbf{x}$

$$\mu \equiv \lim_{t\to 1} \frac{d \ln f(t\mathbf{x})}{d \ln(t)} = \frac{\sum_{i=1}^{n} MP_i x_i}{f(\mathbf{x})}$$

Define output elasticity of input i

$$\mu_i(\mathbf{x}) \equiv \frac{\partial f(\mathbf{x})}{\partial x_i} \frac{x_i}{f(\mathbf{x})} = \frac{MP_i x_i}{f(\mathbf{x})}$$

Thus

$$\mu(\mathbf{x}) = \sum_{i=1}^{n} \mu_i(\mathbf{x})$$

Sometimes it can be written as $\mu^*(y)$, so you can say the technology displays *locally increasing/constant/decreasing return to scale* at output level y.

Short run: MP and AP of variable input

Two input K and L, K is fixed at $\overline{K}$ in the short run.
How does $y = f(L; \overline{K})$ change with L?

- Average product of labour $AP_L = \frac{f(L;\overline{K})}{L}$
- Marginal product of labour $MP_L = \frac{\partial f(L;\overline{K})}{\partial L}$
  - $MP_L \uparrow$ in L when L is small
    - *efficiency gain from the division of labour*
  - $MP_L \downarrow$ in L when L is large
    - *exhaust the benefit from the division of labour and MP_L may become negative due to the constraint on capital input*

MP, AP and TP in the short run

- For the first unit of labour input $MP_L = AP_L$
- How does AP_L change with L when $MP_L > AP_L$?
- How does AP_L change with L when $MP_L < AP_L$?
- Output elasticity of input: $\mu_i(\mathbf{x}) = \frac{MP_i x_i}{f(\mathbf{x})} = \frac{MP_i}{AP_i}$
- What happens to $TP(L; \overline{K})$ when $MP_L = 0$?
- When does MP_L achieve maximum?
- $AP(L; \overline{K})$ and $TP(L; \overline{K})$ curves
- When a production function is concave, there is diminishing MP_L.

Cost minimization

Firm's cost minimization problem (similar to consumers' EMP) - assuming perfectly competitive input markets

$$c(\mathbf{w}, y) \equiv min_{\mathbf{x}}\ \mathbf{w} \cdot \mathbf{x}$$
$$s.t. \quad f(\mathbf{x}) \geq y$$

Assume the Inada conditions so that $lim_{x_i \to 0} f_i = +\infty$, the FOC is

$$\frac{MP_i}{w_i} = \frac{MP_j}{w_j}$$

- Solve for conditional input demand function, $\mathbf{x}^*(\mathbf{w}, y)$, and cost function, $c(\mathbf{w}, y) = \mathbf{w} \cdot \mathbf{x}^*$
- Interpretation of the Lagrange multiplier: $MC(y)$
- Graphical presentation of cost function - isocost lines

Properties of $c(\mathbf{w}, y)$ and $\mathbf{x}^*(\mathbf{w}, y)$

- Cost function $c(\mathbf{w}, y)$
  - Expenditure function
  - $c(\mathbf{w}, y) = 0$ as $f(\mathbf{0}) = 0$
  - Continuous
  - Increasing in $\mathbf{w}$
  - Homogeneous of degree 1 in $\mathbf{w}$
  - Concave in $\mathbf{w}$
  - Shepard lemma: $\nabla_{\mathbf{w}} c(\mathbf{w}, y) = \mathbf{x}(\mathbf{w}, y)$
- Conditional input demand function $\mathbf{x}(\mathbf{w}, y)$
  - Hicksian demand function
  - Homogeneous of degree 0 in $\mathbf{w}$
  - Symmetric and negative semi-definite substitution matrix

Cost function with homothetic production function

When the production function is homothetic, $c(\mathbf{w}, y)$ is multiplicatively separable in input prices and output.
It can be written as $c(\mathbf{w}, y) = h(y)c(\mathbf{w}, 1)$, where $h(y)$ is strictly increasing.

Proof:
$f(x)$ is homothetic $\Rightarrow f(x) = m(g(x))$, homog. $g(.)$ and monotone $m(.)$.
Then

$$c(\mathbf{w}, y) = \min_{\mathbf{x}} \ \mathbf{w} \cdot \mathbf{x} \ s.t. \ m\left(\frac{m^{-1}(1)}{m^{-1}(y)} g(\mathbf{x})\right) \geq 1$$

Cost function with homothetic production function

Let $\widehat{\mathbf{x}} = \frac{m^{-1}(1)}{m^{-1}(y)}\mathbf{x}$,

$$\begin{aligned} c(\mathbf{w}, y) &= \frac{m^{-1}(y)}{m^{-1}(1)} \min_{\widehat{\mathbf{x}}} \ \mathbf{w} \cdot \widehat{\mathbf{x}} \ s.t.\ f(\widehat{\mathbf{x}}) \geq 1 \\ &= \frac{m^{-1}(y)}{m^{-1}(1)} c(\mathbf{w}, 1) = h(y)c(\mathbf{w}, 1) \end{aligned}$$

Homethetic and homogeneous production functions

- When $f(x)$ is homothetic, $\mathbf{x}^*(\mathbf{w}, y)$ are multiplicatively separable in input prices and output.
  It can be written as $\mathbf{x}^*(\mathbf{w}, y) = h(y)\mathbf{x}(\mathbf{w}, 1)$, where $h(y)$ is strictly increasing.
- When the production function is homogeneous of degree $\alpha > 0$,

$$
\begin{aligned}
c(\mathbf{w}, y) &= y^{\frac{1}{\alpha}} c(\mathbf{w}, 1) \\
\mathbf{x}(\mathbf{w}, y) &= y^{\frac{1}{\alpha}} \mathbf{x}(\mathbf{w}, 1)
\end{aligned}
$$

Use the fact that

$$
f(\mathbf{x}) = y \Leftrightarrow f\left(\frac{\mathbf{x}}{y^{\frac{1}{\alpha}}}\right) = 1
$$

Short-run cost functions

A simple example with two inputs: capital K and labour L
Suppose capital is fixed at $\overline{K}$ in the short run

- Labour requirement function: $L(y;\overline{K}) = f^{-1}(y;\overline{K})$
- $\frac{\partial L(y;\overline{K})}{\partial y}$: the reciprocal of $MP_L = \frac{\partial y}{\partial L}$
- $\frac{L(y;\overline{K})}{y}$: the reciprocal of $AP_L = \frac{y}{L}$
- Marginal cost: $MC(y) = w \cdot \frac{\partial L(y;\overline{K})}{\partial y}$
- Variable cost: $VC(y) = w \cdot L(y;\overline{K})$
- Average variable cost: $AVC(y) = w \cdot \frac{L(y;\overline{K})}{y}$
- Total cost: $TC(y) = w \cdot L(y;\overline{K}) + r\overline{K}$
- Average total cost: $ATC(y) = \frac{w \cdot L(y;\overline{K}) + r \cdot \overline{K}}{y}$

From short-run curves to long-run curves

- Short-run cost curves with $\overline{K_1}$, $\overline{K_2}$,...
- Long run cost curve is the lower envelope of the entire family of short-run curves

Suppose (L^*, K^*) is the long-run optimal factor combination to produce y^* given the factor prices. Then L^* is also the required labour input for producing y^* in a short-run situation if the capital input is fixed at K^*

- $STC(y^*; K^*) = LTC(y^*)$, $SAC(y^*; K^*) = LAC(y^*)$
- $STC(y; K^*) > LTC(y)$, $SAC(y; K^*) > LAC(y)$ when $y \neq y^*$
- $SMC(y^*) = LMC(y^*)$
- $LMC(y) < SMC(y; K^*)$ to the right of y^*;
  $LMC(y) > SMC(y; K^*)$ to the left of y^*

From short-run curves to long-run curves

A Numerical Example

$$\min_{K,L} rK + wL + \text{FC}$$

such that

$$K^{\alpha} L^{1-\alpha} \geq y$$

From short-run curves to long-run curves

A Numerical Example

$$\min_{K,L} rK + wL + \text{FC}$$

such that

$$K^{\alpha}L^{1-\alpha} \geq y$$

Let $\alpha = 0.5, r = w = 1$. It is easy to see that

1. $LAC(y) = 2 + \text{FC}/y$
2. $SAC(y) = \overline{K}/y + y/\overline{K} + \text{FC}/y$
3. $MC(y) = 2y/\overline{K}$

From short-run curves to long-run curves

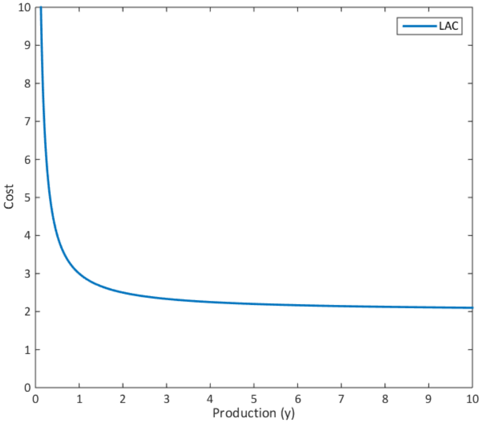

Figure: Cost functions

From short-run curves to long-run curves

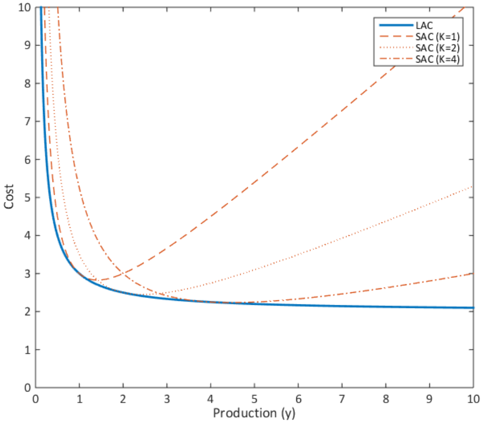

Figure: Cost functions

From short-run curves to long-run curves

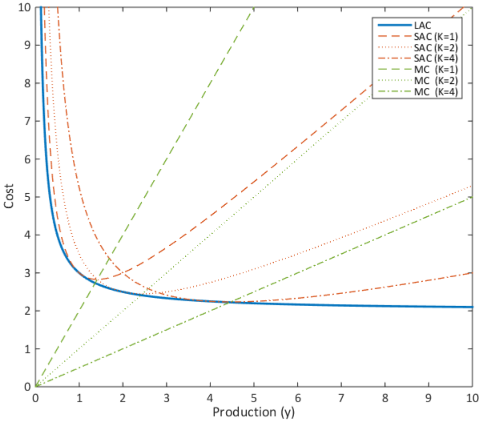

Figure: Cost functions

Comparison in the isoquants diagram

Start from an optimal point in the long run, given $\mathbf{w}$ and y

- To increase output level to y', additional cost in the short run?
- To increase output level to y', additional cost in the long run?
- To reduce output level to y'', cost saving in the short run?
- To reduce output level to y'', cost saving in the long run?

Relationship between *SMC* and *LMC*

Divide the set of inputs $\mathbf{x}$ into variable inputs $\mathbf{x}_v$ (e.g. Labor) and inflexible inputs $\mathbf{x}_f$ (e.g. Capital).

Fixed $\mathbf{x}_f = \bar{\mathbf{x}}_f$ in the short run.

- The short-run cost minimization problem is

$$sc(\mathbf{w}, y, \bar{\mathbf{x}}_f) \equiv \min_{\mathbf{x}_v} \mathbf{w}_v \cdot \mathbf{x}_v + \mathbf{w}_f \cdot \bar{\mathbf{x}}_f \ s.t. \ f(\mathbf{x}_v; \bar{\mathbf{x}}_f) \geq y$$

$ET : \frac{\partial sc}{\partial x_j} = w_j - SMC(y)MP_j, \ \forall x_j \in \mathbf{x}_f$

$FOC : w_j = SMC(y)MP_j, \ \forall x_j \in \mathbf{x}_v$

Relationship between *SMC* and *LMC*

In the long-run

- The long-run problem is

$$c(\mathbf{w}, y) \equiv \min_{\mathbf{x}_v, \mathbf{x}_f} \mathbf{w}_v \cdot \mathbf{x}_v + \mathbf{w}_f \cdot \mathbf{x}_f \ s.t.\ f(\mathbf{x}_v, \mathbf{x}_f) \geq y$$

$FOC: w_i = LMC(y)MP_i, \ \forall x_i \in \mathbf{x} = \{\mathbf{x}_v, \mathbf{x}_f\}$

- The long-run problem is equivalent to

$$c(\mathbf{w}, y) \equiv \min_{\mathbf{x}_f} \ sc(\mathbf{w}, y, \mathbf{x}_f)$$

$FOC: \frac{\partial sc}{\partial x_j} = 0 = w_j - SMC(y)MP_j, \ \forall x_j \in \mathbf{x}_f$

$\Rightarrow w_j = SMC(y)MP_j$

Thus $SMC(y) = LMC(y)$ at $(\mathbf{x}_v^*, \mathbf{x}_f^*)$

Comparative statics - Cost-sensitivity

Three firms all use labour L and capital K in their production. They have different technologies

- Firm A has a Leontief production function
- Firm B has a linear production function
- Firm C has a Cobb-Douglas production function

At the initial point, they incur the same total costs in producing a given amount of output.

There is a sudden increase of 10% in the capital rental price r. To produce the same level of output, how would their total costs change? Rank the three firms by the change in their total costs.

Application: recovering market power

De Loecker & Warzynski (*AER*, 2012)

- Suppose a firm has market power on the output market and is a price taker on the input market of factor i - $w_i(y) = w_i$. Let the inverse demand for its output be $p(y)$.
- Market power can be measured by markup $\frac{p}{MC}$. If a firm is a cost minimizer, then its market power can be recovered from production information.

$$\begin{aligned} \text{Cost minimization} \quad &\Rightarrow \quad w_i = MC(y) \cdot MP_i(y) \\ &\Rightarrow \quad \frac{p(y)}{MC(y)} = \frac{p(y)MP_i(y)}{w_i} = \frac{\mu_i}{\alpha_i} \end{aligned}$$

where μ_i is the output elasticity of factor i and α_i is its share of expenditure in total revenue.

Duality: $f(\mathbf{x}) \Leftrightarrow c(\mathbf{w}, y)$ and $\mathbf{x}(\mathbf{w}, y)$

- Recover production function $f(\mathbf{x})$ from $c(\mathbf{w}, y)$

$$\begin{aligned} f(\mathbf{x}) &\equiv \max\{y \geq 0 | \mathbf{w}\cdot\mathbf{x} \geq c(\mathbf{w}, y), \forall \mathbf{w} \gg 0\} \\ &= \max\left\{y \geq 0 | y \leq c^{-1}(\mathbf{w}, \mathbf{w}\cdot\mathbf{x}), \forall \mathbf{w} \gg 0\right\} \end{aligned}$$

1. Start with input $\mathbf{x}$
2. For any factor price vector $\mathbf{w} \gg 0$, find all the output levels that can be produced with budget $\mathbf{w}\cdot\mathbf{x}$, denote the set by $\mathbf{Y}_{\mathbf{w},\mathbf{x}}$

$$\mathbf{Y}_{\mathbf{w},\mathbf{x}} \equiv \{y : c(\mathbf{w}, y) \leq \mathbf{w}\cdot\mathbf{x}\}$$

3. Construct the intersection of all these output sets

$$\mathbf{Y}_{\mathbf{x}} \equiv \bigcap_{\mathbf{w}} \mathbf{Y}_{\mathbf{w},\mathbf{x}}$$

4. Find the largest element in

$$f(\mathbf{x}) = y = \max \mathbf{Y}_{\mathbf{x}}$$

Duality: $f(\mathbf{x}) \Leftrightarrow c(\mathbf{w}, y)$ and $\mathbf{x}(\mathbf{w}, y)$

- Derive from input demand $\mathbf{x}(\mathbf{w}, y)$ to $f(\mathbf{x}) = y$
  - If $x_i(\mathbf{w}, y)$ is homogeneous of degree 0 and $\left(\frac{\partial x_i}{\partial w_j}\right)$ is a symmetric negative semi-definite matrix.
  - Then $\sum_{i=1}^{n} w_i x_i(\mathbf{w}, y)$ has all the properties of a cost function
  - So we can use this cost function to reconstruct the original technology
  - The proposition is called "Integrability for Cost Functions".

Example

- What technology can generate cost function

$$c(\mathbf{w}, y) = y w_1^{\alpha} w_2^{1-\alpha}$$

  - Conditional factor demand

$$\begin{aligned} x_1 &= \alpha y w_1^{\alpha-1} w_2^{1-\alpha} \\ x_2 &= (1-\alpha) y w_1^{\alpha} w_2^{-\alpha} \end{aligned}$$

  - Try to get rid of **w**

$$\left(\frac{x_1}{\alpha y}\right)^{\alpha} = \left(\frac{x_2}{(1-\alpha)y}\right)^{\alpha-1} \Rightarrow y = \frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} x_1^{\alpha} x_2^{1-\alpha}$$

- Exercises: Recover the technology that generates

$$c(w_1, w_2, y) = \left(\left(\frac{w_1}{\alpha_1}\right)^r + \left(\frac{w_2}{\alpha_2}\right)^r\right)^{\frac{1}{r}} y$$

The CES production function

Isocost Curves in the Factor Price Space

Isocost curve in the **w** space:

$$c(\mathbf{w}, y) = \overline{c}$$

Differentiate two isocost curves: $c(\mathbf{w}, y) = \overline{c}$ and $\mathbf{w} \cdot \mathbf{x} = \overline{c}$.

Take total derivative to find the slope $\frac{\Delta w_2}{\Delta w_1}$

$$\left|\frac{\Delta w_1}{\Delta w_2}\right| = \frac{\partial c/\partial w_2}{\partial c/\partial w_1} = \frac{x_2^*}{x_1^*}$$

Slope of isoquants at the cost minimizing point in the input space

$$\left|\frac{\Delta x_2}{\Delta x_1}\right| = \frac{\partial f/\partial x_1}{\partial f/\partial x_2} = \frac{w_1}{w_2}$$

Curvatures of Isoquants and Isocost curves

- small curvature of isoquants (linear production function)
  $\Rightarrow$ strong substitutability
  $\Rightarrow$ change in factor price $\rightarrow$ big adjustment in factor usage
  $\Rightarrow$ big curvature of isocost curves
- big curvature of isoquants (Leontief production function)
  $\Rightarrow$ weak substitutability
  $\Rightarrow$ change in factor price $\rightarrow$ small adjustment in factor usage
  $\Rightarrow$ small curvature of isocost curves

Cost Function and Returns to Scale

Elasticity of scale at cost minimizing point $\mathbf{x}^*$

$$
\begin{aligned}
\mu(\mathbf{x}^*) &= \sum_{i=1}^{n} \frac{MP_i x_i}{f(\mathbf{x})} \\
&= \sum_{i=1}^{n} \frac{w_i x_i}{MC(y) f(\mathbf{x})} \\
&= \frac{c(\mathbf{w}, y)}{MC(\mathbf{w}, y) f(\mathbf{x})} \\
&= \frac{AC(\mathbf{w}, y)}{MC(\mathbf{w}, y)}
\end{aligned}
$$

- Increasing local returns to scale $\Leftrightarrow AC > MC$
- Constant local returns to scale $\Leftrightarrow AC = MC$
- Decreasing local returns to scale $\Leftrightarrow AC < MC$

Profit maximization problem

Assume the output market is perfectly competitive, use $c(\mathbf{w}, y)$

$$\begin{aligned} \max_{y \geq 0} \quad & py - c(\mathbf{w}, y) \\ FOC: \quad & p = MC(y^*) \\ SOC: \quad & \frac{d^2 c(\mathbf{w}, y)}{dy^2}\Big|_{y=y^*} \geq 0 \end{aligned}$$

An alternative method,

$$\begin{aligned} \max_{\mathbf{x} \geq 0} \quad & pf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x} \\ FOC: \quad & pMP_i - w_i = 0 \ for \ \forall i = 1, ..., n \\ SOC: \quad & \left(\frac{\partial MP_i}{\partial x_j} \right) \text{negative semi-definite} \end{aligned}$$

- Both FOCs implies $MC = \frac{w_i}{MP_i}$
- SOCs - convexity of cost function (in y) and concavity of production function (in $\mathbf{x}$)

SOCs

Concave production function implies convex cost function.

- Given $\mathbf{w}$, $\mathbf{x}$ and $\mathbf{x}'$ are the cost-minimizing input vectors for producing y and y'. Thus

$$tc(\mathbf{w}, y)+(1-t)c(\mathbf{w}, y') = t\mathbf{w}\cdot\mathbf{x}+(1-t)\mathbf{w}\cdot\mathbf{x}' = \mathbf{w}\cdot(t\mathbf{x} + (1-t)\mathbf{x}')$$

- $f(.)$ is concave $\Rightarrow ty + (1-t)y' \leq f\left(t\mathbf{x} + (1-t)\mathbf{x}'\right)$
- $c(\mathbf{w}, y)$ is non-decreasing in y. Thus

$$\begin{aligned} c\left(\mathbf{w}, ty + (1-t)y'\right) &\leq c\left(\mathbf{w}, f\left(t\mathbf{x} + (1-t)\mathbf{x}'\right)\right) \\ &\leq \mathbf{w}\cdot\left(t\mathbf{x} + (1-t)\mathbf{x}'\right) \\ &= tc(\mathbf{w}, y) + (1-t)c(\mathbf{w}, y') \end{aligned}$$

Interpretation: concave production function means given an increment in input, the increment in output is less than that of a linear function. In other words, an increment in output requires a more than linear increment in input. Thus the increment in cost is more than linear.

Profit function

Solution to profit maximization problem (if exists)

- Output supply function: $y^*(p, \mathbf{w})$
- Input demand function: $\mathbf{x}^*(p, \mathbf{w})$
- Profit function: $\pi(p, \mathbf{w}) = py^*(p, \mathbf{w}) - \mathbf{w} \cdot \mathbf{x}^*(p, \mathbf{w})$

It is necessary that SOCs are satisfied to make maximum exist.

- Counter-example: increasing returns to scale technology

Properties of $\pi(p, \mathbf{w})$

- Increasing in p
- Decreasing in $\mathbf{w}$
- Homogeneous of degree 1 in $(p, \mathbf{w})$
- Hotelling's lemma (Envelope Theorem)

$$\begin{aligned} \frac{\partial \pi(p, \mathbf{w})}{\partial p} &= y^*(p, \mathbf{w}) \\ -\frac{\partial \pi(p, \mathbf{w})}{\partial w_i} &= x_i^*(p, \mathbf{w}) \textit{ for } \forall i = 1, ..., n \end{aligned}$$

- Convex in $(p, \mathbf{w})$

Convexity of profit function

$$
\begin{aligned}
\pi(p^t, \mathbf{w}^t) &= p^t y^t - \mathbf{w}^t \cdot \mathbf{x}^t \\
&= \left(tp + (1-t)p'\right) y^t - \left(t\mathbf{w} + (1-t)\mathbf{w}' \cdot\right) \mathbf{x}^t \\
&= t(py^t - \mathbf{w} \cdot \mathbf{x}^t) + (1-t)\left(p'y^t - \mathbf{w}' \cdot \mathbf{x}^t\right) \\
&\leq t(py - \mathbf{w} \cdot \mathbf{x}) + (1-t)\left(p'y' - \mathbf{w}' \cdot \mathbf{x}'\right) \\
&= \pi(p, \mathbf{w}) + (1-t)\pi(p', \mathbf{w}')
\end{aligned}
$$

Because of the maximization process,

- when there is a decrease in $\mathbf{w}$(or an increase in p), the profit is going to increase at least as fast as a linear function
- when there is an increase in $\mathbf{w}$ (or a decrease in p), the profit is going to decrease at most as fast as a linear function
- The Hessian matrix is positive semi-definite

Properties of $y^*(p, \mathbf{w})$ and $x_i^*(p, \mathbf{w})$

- Homogeneity of degree 0 in $(p, \mathbf{w})$
- Output supply: non-negative own price effects on profit

$$\frac{\partial y^*(p, \mathbf{w})}{\partial p} = \frac{\partial^2 \pi(p, \mathbf{w})}{\partial p^2} \geq 0$$

- Input demand: non-positive own price effects on profit

$$\frac{\partial x_i^*(p, \mathbf{w})}{\partial w_i} = -\frac{\partial^2 \pi(p, \mathbf{w})}{\partial w_i^2} \leq 0 \textit{ for } \forall i = 1, ..., n$$

- Substitution matrix: Hessian matrix of the profit function - symmetric and positive semi-definite

Short-run Profit Maximization

- Fixed costs: do not vary with output level
  - Sunk fixed costs: predetermined and cannot be changed *no matter* $y = 0$ *or* $y > 0$
  - Non-sunk fixed costs: not incurred if $y = 0$
- Total Costs and average costs
  - $STC = FC + TVC = (SC + \underbrace{NSFC) + TVC} = SC + NSC$
  - $SAC = AFC + AVC = (ASC + \underbrace{ANSFC) + AVC} = ASC + ANSC$
- $\pi = TR - STC = \underbrace{TR - NSC} - SC = PS - SC$
- Conditional on operating: $p = MC(y)$
- When to shut down?

Short-run Profit Maximization

1. $P_1 = MC(q_1^*) < AVC(q_1^*) \Rightarrow TR(q_1^*) < TVC(q_1^*)$
   Shut down because of negative surplus
2. $AVC(q_2^*) < P_2 = MC(q_2^*) < ANSC(q_2^*) \Rightarrow$
   $TR(q_2^*) < NSC(q_2^*) = NSFC + TVC(q_2^*)$
   Shut down because of negative surplus
3. $P_3 = MC(q_3^*) = ANSC(q_3^*)$, thus the minimum point of $ANSC$
   $\Rightarrow TR(q_3^*) = NSC(q_3^*) = NSFC + TVC(q_3^*)$
   Shut down or run (with negative profit and zero suplus)
4. $ANSC(q_4^*) < P_4 = MC(q_4^*) \leq SAC(q_4^*) \Rightarrow$
   $NSC(q_4^*) = NSFC + TVC(q_4^*) < TR(q_4^*) < STC(q_4^*)$
   Run (with negative profit but positive surplus)
5. $P_5 = MC(q_5^*) > SAC(q_5^*) \Rightarrow TR(q_5^*) > STC(q_5^*)$
   Run (with positive profit)