Game Theory 106G
Key Concepts of Game Theory

Menghan Xu

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What is Game Theory?

Definition

Game theory is a tool to analyze interaction among a group of rational agents who behave strategically. A more descriptive term would be "Interactive Decision Theory".

Normal-Form Games

Definition

A normal-form game is defined by

1. A set of *players:* $I = \{1, 2, 3, \cdots, n\}$.
2. *A set of actions* for each player: A_i, $i \in I$.
3. *Payoff (= Utility) for each player for each action profile* $a = (a_1, ..., a_n) = (a_i, a_{-i}) : u_i(a_i, a_{-i}), i \in I$ where $a_{-i} = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$ is a short notation for others' actions. In general, "$-i$" will refer to player i's opponents

Normal-form games are used to describe simultaneous games with complete information

Example

In the oligopoly game, the set of players are the firms; the set of actions were the choices of quantities or prices; and the payoffs were the profits perceived by each firm.

- Cournot competition;
- Bertrand competition

Example

Two-player zero-sum games are such that $\sum_{i=1}^{2} u_i(a) = 0$. In other words, whatever one player wins the other loses

- Gambling
- Paper-Scissor-Rock

Payoff Matrix

- When there are two players and the set of strategies is finite the game can be represented by matrices.
- Rows represent strategies of player 1, whereas columns represent strategies of player 2.
- The first entry in each box is player 1's payoff for the corresponding strategy profile; the second is player 2's.

Prisoner's Dilemma

Example

Two prisoners are charged of a serious crime. If they both *C*ooperate and do not confess, they will be released after one year of investigation. If they both *D*efect and confess, they will both be sent to prison for three years. If only one of them *D*efects but the other one *C*ooperates, the defector gets out of prison for free, while the cooperator receives a sentence of five years. Every prisoner wants to minimize her own time in prison.

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{2} \\
 & & C & D \\
\hline
1 & C & -1\ ,\ -1 & -5\ ,\ 0 \\
\hline
 & D & 0\ ,\ -5 & -3\ ,\ -3 \\
\hline
\end{array}
$$

Pure strategies

Definition

A pure strategy of player i is an action profile which specify which action to take facing each situations.

- In a one-shot game,(each player only move once), set of pure strategy is equivalent to set of action.
- In sequential games or repeated games, strategy and action are not equivalent to each other!

- What should you do as player 1?
- If $s_2 = C$, then your payoffs are
$$\left\{\begin{array}{l} s_1 = C, u_1(C, C) = -1 \\ s_1 = D, u_1(D, C) = 0 \end{array}\right. \Rightarrow \text{Choose } D.$$
- If $s_2 = D$, then your payoffs are
$$\left\{\begin{array}{l} s_1 = C, u_1(C, D) = -5 \\ s_1 = D, u_1(D, D) = -3 \end{array}\right. \Rightarrow \text{Choose } D.$$

- Therefore, no matter what your opponent chooses, you are always better off by playing D.
- In this case, we will say that D strictly dominates C.
- The game is symmetric, we can conduct the same analysis for player 2. Then we should expect that the equilibrium strategy to this game is (D, D)

Mixed strategies

Definition

A mixed strategy of player i is a probability distribution σ_i over her actions a_i, formally $\sigma_i \in \Delta\left(A_i\right)$.

- A probability is a number between 0 and 1, and the sum of probabilities must add up to 1.
- The greater the probability, the greater the chance the individual plays that strategy

- For example, in the prisoner's dilemma, a mixed strategy for player 1 is defined by numbers $\sigma_1(C), \sigma_1(D) \in [0,1]$ such that $\sigma_1(C) + \sigma_1(D) = 1$
- For instance $(1/2, 1/2)$ means that each strategy is played with equal chance, in other words the player first toss a coin before deciding which strategy she is going to play
- Note that a (pure) strategy is just a mixed strategy that assigns probability 1 to such strategy.

Definition

A belief of player i is a probability distribution $\sigma_{-i} \in \Delta(A_{-i})$ over strategies a_{-i} by all other players.

Definition

A mixed strategy profile σ is defined as
$\sigma = (\sigma_1, ..., \sigma_n) = (\sigma_i, \sigma_{-i})$

- Because players randomize on their actions, there will be multiple outcomes of the a game. How do we evaluate the value of mixed strategies?

Expected Value

- Consider a random variable X. This random variable is equal to x_i with probability p_i for $i = 1, ..., n$ as summarized in the following table:

p_1	p_2	$\cdots$	$\cdots$	p_n
x_1	x_2	$\cdots$	$\cdots$	x_n

.

- Then the expected value (mean) of this random variable is defined by

$$E\left[X\right] = p_1 x_1 + p_2 x_2 + ... + p_n x_n$$

- So the expected value of payoffs would be used in analyzing mixed strategies.

- Note that the probability of the realization of a action profile $a = (a_1, ..., a_n)$ is just $\prod_{j=1}^{n} \sigma_j (a_j)$ since it is assumed that the mixed strategies of each agent are independent

Example

The probability of having two tails when a coin is tossed twice is just $1/2 * 1/2 = 1/4$

Example

Consider the prisoner's dilemma and suppose that $\sigma_1 = (p, 1-p)$ and $\sigma_2 = (q, 1-q)$. The following table summarizes the probability of being in each strategy profile

	C	D
C	pq	$p(1-q)$
D	$(1-p)q$	$(1-p)(1-q)$

Note that the sum of all of them adds up to 1 and each is between 0 and 1

-

- Therefore, given his payoff function $u_i(a_i, a_{-i})$ and his beliefs σ_{-i} of the strategy choice of other players, the rational player i wants to maximize his expected utility:

$$\text{choose } \sigma_i \text{ to maximize } U_i(\sigma_i, \sigma_{-i})$$

where U_i denotes the Expected (or Von Neuman-Morgenstern) Utility and is defined as

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{s \in S} \left(\prod_{j=1}^{n} \sigma_j(s_j) \right) u_i(s)$$

Example

Consider the prisoner's dilemma and suppose that player 1 believes that Player 2 will play C with probability q and D with probability $1-q$, that is $\sigma_2 = (q, 1-q)$

- If player 1 plays C her payoff will be $U_1(C, \sigma_2) = q(-1) + (1-q)(-5) = -5 + 4q$.
- On the other hand, if player 1 plays D her payoff will be $U_1(D, \sigma_2) = q(0) + (1-q)(-3) = -3 + 3q$.

- Note that $U_1(C, \sigma_2) > U_1(D, \sigma_2)$ if and only if $q > 2$, which is not possible
- Therefore player 1 will prefer to play D given any mixed strategy of player 2
- It should not be surprising that if D strictly dominated C in pure strategies, then it will also strictly dominate any mixed strategy, since mixed strategies imply the use of the strictly dominated strategy

Example

Consider now a variation of the prisoner's dilemma

- In the variation of the prisoners' dilemma, prisoner 1 prefers to cooperate with 2 if 2 also cooperates but not if 2 defects

1 \ 2	C	D
C	-1 , -1	-5 , 0
D	-2 , -5	-3 , -3

- As prisoner 2's payoffs have not changed, it is still the case that D strictly dominates C for player 2.

Example

- The situation is more difficult for player 1 :
- If $s_2 = C$, then 1's payoffs are
  $\begin{cases} s_1 = C, u_1(C, C) = -1 \\ s_1 = D, u_1(D, C) = -2 \end{cases} \Rightarrow$ Choose C.
- If $s_2 = D$, then 1's payoffs are
  $\begin{cases} s_1 = C, u_1(C, D) = -5 \\ s_1 = D, u_1(D, D) = -3 \end{cases} \Rightarrow$ Choose D.

Example

- What prisoner 1 should do thus depends on what he believes about the strategy of prisoner 2.
- Using the same beliefs as before, his expected payoff from choosing C himself is $-1q - 5(1-q) = -5 + 4q$ whereas D gets him $-2q - 3(1-q) = -3 + q$ and he prefers C over D whenever $-5 + 4q \geq -3 + q$ which is the case whenever $q \geq \frac{2}{3}$.
- Notice that, in first version of prisoners' dilemma, no matter what player 2 does(pure strategies or mixed strategy), player 1 always prefers to defect. But in the second version, player 1's preference depends on player 2's action.

Strict Dominance

Definition

Strategy s_i strictly dominates s_i' if for EVERY strategy profile of the others $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$, i strictly prefers s_i over s_i'

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

- In this case we also say that s_i' is strictly dominated by s_i.

Definition

If s_i strictly dominates ALL other strategies of player i, then we say s_i is **the strictly dominant** strategy for player i.

- In the Prisoner's dilemma D strictly dominates C for both players, therefore D is the strictly dominant strategy for both players.
- In a game, if each player has strictly dominant strategy, there is a unique solution to the game - all players choose their strict dominant strategies.

In most situations, not every player has strictly dominant strategy, but as long as some of them have, the analysis can be simplified.

Example

Consider the modified prisoner's dilemma

		2: C	2: D
1	C	$-1\ ,\ -1$	$-5\ ,\ 0$
	D	$-2\ ,\ -5$	$-3\ ,\ -3$

- As discussed before, player 1 does not have strictly dominant strategy. But D is still player 2's strictly dominant strategy.
- To solve the game, we can first pin down player 2's strategy and then analyze player 1's.
- Since player 1 knows that player 2 will always choose D, he will choose D as well.

- If D strictly dominates C, conversely speaking, we can say C is strictly dominated by D. In this situation, a rational player never chooses C.
- But in order to be strictly dominated, a strategy does not necessarily have to be dominated by another **pure** strategy

Definition

A pure strategy s_i is strictly dominated if there is a (mixed) strategy σ_i such that $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$ for all strategy profiles s_{-i}

Example

Consider the following game

		B		
		C	X	D
A	C	2, 3	1, 1	0, 0
	D	3, 0	0, 1	1, 3

Example

- At first sight, there is no single pure strategy that can be strictly dominated by another **pure** strategy.
- However, note that the mixed strategy that puts a 0.5 probability on playing C and 0.5 probability on D strictly dominates X
  - When player A plays C, player B's utility of playing X is 1. The expected payoff of playing $(0.5, 0.5)$ of (C, D) is

$$0.5 \times 3 + 0.5 \times 0 = 1.5$$

    which is larger than 1.
  - Similarly, when player A plays D,player B's utility of playing X is 1. The expected payoff of playing $(0.5, 0.5)$ of (C, D) is

$$0.5 \times 0 + 0.5 \times 3 = 1.5$$

    which is larger than 1.

Example

- In conclusion, regardless which strategy player A takes, X is strictly dominated by the mixed strategy.
- By definition, X is a strictly dominated strategy.

- Note that, there are more than one mixed strategy that strictly dominates X. As long as you find one of them, X is a strictly dominated strategy.
- If a strategy is strictly dominated, we can eliminate it when we analysis the game. Since anyway, this strategy will never be chosen if player is rational.

Example

- Therefore, we can reduce the game to

		B	
		C	D
A	C	2 , 3	0 , 0
	D	3 , 0	1 , 3

- Now for player A, C is strictly dominated by D.

Example

- Therefore, we can reduce the game to

		B	
		C	D
A	D	3 , 0	1 , 3

- Finally, D strictly dominates C for player B. Therefore, the game is left with

		B	
		D	
A	D	1	3

- We can solve this game by iteratively deleting strictly dominated actions.

1. $X <_B (0.5C, 0.5D)$. Delete $s_B = X$.
2. $C <_A D$. Delete $s_A = C$.
3. $C <_B D$. Delete $s_B = C$.

- And we are left with only one action profile, (D, D).

Iterated Strict Dominance

Definition

The process of iterated deletion of strictly dominated strategies proceeds as follows

- Set $S_i^0 = S_i$ i.e all the (pure) strategies and $\Sigma_i^0 = \Sigma_i$, all the mixed strategies.
- Now define recursively S_i^n as the set of strategies that are not strictly dominated by any strategy belonging to $\Sigma_i^{n-1} = \Delta\left(S_i^{n-1}\right)$, which is the set of mixed strategies that uses the strategies in S_i^{n-1}
- Of course it is the case that $S_i^n \subseteq S_i^{n-1}$
- Set $S_i^\infty = \lim_{n\to\infty} S_i^n$ as the set of pure strategies that survive iterated deletion of strictly dominated strategies

Example

Consider the following game

		B		
		C	X	D
A	C	2, 3	1, 1	0, 0
	D	3, 0	0, 1	1, 3

- $S_A^0 = \{C, D\}$ and $S_B^0 = \{C, X, D\}$
- Only $s_B = X$ is deleted in first round, so $S_A^1 = \{C, D\}, S_B^1 = \{C, D\}$
- Next, $s_A = C$ is deleted in second round, so $S_A^2 = \{D\}, S_B^2 = \{C, D\}$
- Next, $s_B = C$ is deleted in second round, so $S_A^3 = \{D\}, S_B^3 = \{D\}$
- Since each player has only one strategy left, the iteration stops here.

A Beauty Contest

Example

Each student chooses a number between 1 and 100. The student whose number is closest to 1/2 of the average of the chosen numbers wins the game. Denote the strategies of the players in the game by $s_i \in [1, 100]$ and assume that the payoff of a player who guessed $s_i = x$ is given by $-\left|x - \frac{1}{2}\bar{s}\right|$ where $\bar{s} = \frac{1}{n}\sum_{i=1}^{n} s_i$.

- How should one play this game? Can we apply iterated deletion?

- For example there is no point on choosing a 100 since even if everybody chooses a 100 half of the average is at most 50. Alternatively speaking, 100 is strictly dominated by 50.
- The reasoning could be applied to all strategies $s_i' \in (50, 100]$ since half of the average is always a number smaller than 50.
- Since all players are rational, then every player knows that nobody will choose $s_i' > 50$

- Therefore, in the first round of deletion we can remove that first set of strictly dominated strategies for every player, that means $S_i^1 = [1, 50]$
- But if nobody will choose any number greater than 50, then half of the average is never greater than 25
- Therefore, by the same reasoning, in a second round of deletion everybody will remove any strategy $s_i' > 25$, that means $S_i^2 = [1, 25]$

- But again this means half of the average is going to be smaller than 12.5, and in a third round everybody removes $s_i''' > 12.5$, which means $S_i^3 = [1, 12.5]$
- You can continue this reasoning and you will notice that in the limit $S_i^\infty = \{1\}$
- That means everybody picks $s_i = 1$, and that constitutes the equilibrium

Definition

A game is called *dominance solvable* if each player has exactly one surviving strategy s_i after the iterated deletion of

- This is the case of the games we have examined until now, but it is not the case in general
- Note that the iteration uses the removal of **strictly** dominated strategies, not weakly

Comment

- Iterated elimination of strictly dominant strategies relies heavily in the concept of common knowledge of rationality.
- But in the real world we are not always sure of how rational our opponents are, which makes us to be more careful before eliminating strategies.

Weak Dominance

There is a another group of concepts.

Definition

Strategy s_i weakly dominates s_i' if for EVERY strategy profile of the others $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$, i weakly prefers s_i over s_i'

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

Definition

f s_i strictly dominates ALL other strategies of player i, then we say s_i is **the weakly dominant** strategy for player i.

Definition

A pure strategy s_i is weakly dominated if there is a (mixed) strategy σ_i such that $u_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for all strategy profiles s_{-i}

Examples

		2	
		C	D
1	C	$0\ ,\ 0$	$-5\ ,\ 0$
	D	$0\ ,\ -5$	$-3\ ,\ -3$

C is weakly dominated by D for both players.

- Weak dominance is weaker than strict dominance.
- We cannot eliminate weakly dominated strategy since we may delete some equilibrium solutions.
- Weakly dominant strategy may not imply unique solution.
- We will see why in next chapter when we formally introduce Nash equilibrium.

Best Response

- Whereas strict dominance relies on the observation of the strategies that will never be played by an agent, one can also ask a complementary question: what are all the strategies that a rational player would play given others' strategies?

Definition

For payoff function $u_i(s_i, s_{-i})$, strategy σ_i is a best-response to beliefs σ_{-i} if

$$U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i}) \text{ for all strategies } \sigma_i' \in \Delta(S_i).$$

The set of all best responses is denoted by $BR_i(\sigma_{-i})$.

Example

Consider the modified of the Prisoners' Dilemma

- The best response set for prisoner B is just $BR_2(p) = D$ for any $p = \sigma_1(D)$ since D is the dominant strategy for player B
- However, what prisoner A should do depends on what he believes about the strategy of prisoner 2.
- If he believes player 2 will play C with probability q, then his expected payoff from choosing C himself is $-1q - 5(1-q) = -5 + 4q$ whereas D gets him $-2q - 3(1-q) = -3 + q$
- He prefers C over D whenever $-5 + 4q \geq -3 + q$ which is the case whenever $q \geq \frac{2}{3}$.
- Alternatively, we say C is the best response to $\sigma_B = (q, 1-q)$ if $q \geq \frac{2}{3}$

- When $q > \frac{2}{3}$, the best response of A is C.
- When $q < \frac{2}{3}$, the best response of A is D.
- When $q = \frac{2}{3}$, the expected payoffs of playing C and D are the same. Therefore, both C and D are best responses.
- In addition, any mixed strategies on C and D are best responses.

Strict Dominance and Best Responses

- A strictly dominated strategy can never be a best response.
- *However, if a strategy is never a best response, it may not be strictly dominated.
- *Can you come up with an example? Hint: consider a game with more than 2 players.

Rationalizability

- We can create a similar argument as the iterated deletion, but with best responses instead of strictly dominated strategies
- Let $BR_i^0 = S_i$ be the set of all pure strategies of player i. (Let's ignore mixed strategy in this section).
- Let $BR_i^1 \subseteq A_i$ be the set of all actions of player i that are a best response for some action profile in the original game.
- ...
- Let $BR_i^{n+1} \subseteq BR_i^n$ be the set of all actions of player i that are a best response in the game with actions BR_j^n

Definition

An strategy s_i is **rationalizable** if it belongs to the set $BR_i^{\infty} = \lim_{n\to\infty} BR_i^n$

- Since best responses are a subset of the strategies that are not strictly dominated, rationalizability generates a set of actions contained in the set of actions generated by the iterated strict dominance
- In two-person games, the two sets induced by strict dominance iteration and best response iteration coincide
- *In genaral

$$BR_i^{\infty} \subseteq S_i^{\infty}$$

Electoral Competition

- What factors determine policy proposals before elections?
- These decisions are strategically interrelated because the number of votes depends not only on one's own proposal but also on others.
- One way to think about this competition is that career politicians choose proposals that will get themselves elected, rather than proposals that represent their own preferred policies.

- Players: $i = R, D$. Republicans and Democrats
- Strategy Sets: each party chooses one policy from nine options $A_R = A_D = \{1, 2, 3, 4, ..., 9\}$. The closer the numbers, the similiar the policies..

- Payoffs:
  - For each policy, 1,2,...,9, there are 100 voters to support it.
  - Each voter votes for the party whose policy is closest to the policy he or she supports.
  - Assume that each party only cares about the total votes it gets, not policy itself. For example, if R chooses 6 and D chooses 3, then the payoffs (=the number of votes) are given by $u_R(6,3) = 500$, $u_D\,(3,6) = 400$.
  - If both parties are equally attractive to a group of voters, then the group is split equally. For example, if R chooses 1 and D chooses 9, then each gets 450 votes, i.e. the voters who support 5 split between R and D evenly.

- Lets first calculate the best response of Democrats when facing the different strategies of the Republicans
    - If republicans choose 1, D will choose 2
    - If republicans choose 2, D will choose 3
    - If republicans choose 3, D will choose 4
    - If republicans choose 4, D will choose 5

-
  - If republicans choose 5, D will choose 5
  - If republicans choose 6, D will choose 5
  - If republicans choose 7, D will choose 6
  - If republicans choose 8, D will choose 7
  - If republicans choose 9, D will choose 8

- Therefore, $BR_D^1 = \{2, ..., 8\}$
- And $BR_R^1 = \{2, ..., 8\}$ by a similar argument
- Another iteration using the same reasoning will lead us to $BR_D^2 = \{3, ..., 7\} = BR_R^2$
- A third round of deletion of strategies will lead us to $BR_D^3 = \{4, ..., 6\} = BR_R^3$

- A final iteration gives us the solution $BR_D^4 = \{5\} = BR_R^4$; in other words, $s_R = s_D = 5$ are the only rationalizable strategies
- This means that both parties will choose the same policy and will tie in the election: a policy convergence result.
- A more formal version of this result is known as the **mean voter theorem**, and it states that in electoral competitions the parties will choose the most preferred policy of the median voter

- A more general version of this game is used to analize location problems that involve:
  - Physical distance of stores when individuals care about costs of transportation
  - Product differentiation when individuals care about how different is the product to what they desire
- This general model is due to Hotelling (1929) and the solution is given when firms choose the same location or product that the mean buyer prefers.

Game Theory 106G
Nash Equilibrium

Menghan Xu

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Refresh

- Ways of solving games:
  - Iterated elimination of strictly dominated strategies
  - Rationalizability by iterating over best responses
- A strategy is strictly dominated if there is another strategy that yields strictly higher payoff given all possible strategy profiles of the other players.
- An action is a best response if there exists a belief over the opponents' strategy profiles such that it gives a highest payoff among all the possible strategies.

- For two player games, the rationalizable strategies coincides with the set of strategies that survive the iterated deletion of strictly dominated strategies
- In more general games, the former is a subset of the latter
- We applied this concepts to a beauty contest and to electoral competition (mean voter theorem)

- Unfortunately, rationalizability gives very weak predictions since it may not eliminate certain outcomes that are not likely to arise in equilibrium

Example

The battle of the sexes is a game in which a couple has to decide whether to see a movie or go to the opera. The girl wants to go to the opera and the boy prefers a movie, but both prefer to go together rather than being alone. Their payoffs are summarized in the following matrix

		Boy	
		O	M
Girl	O	3 , 1	0 , 0
	M	0 , 0	1 , 2

Similar games are called "coordination games" since the best outcomes are obtained when players choose the same action

- As you can see, you cannot eliminate any strategy by either Iterated elimination of strictly dominated strategies or Rationalizability by iterating over best responses.
- We need a more powerful concept that allows us to find a solution to the game.

Nash Equilibrium

Definition

A strategy profile $s^* = (s_1^*, ..., s_n^*)$ is a **pure** Nash equilibrium (NE) if each player's strategy is an optimal response to the other player's strategies, formally $s_i^* \in BR_i\left(s_{-i}^*\right)$, that is

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*) \text{ for all } s_i' \in S_i.$$

*A pure NE is said to be strict if the inequality holds strictly

- As the definition states, the best way to find a pure NE is to fix the strategy of one player and determine the best response to such strategy, then look for profiles where the best responses coincide
- Let's call the method: "double underline algorithm"

Example

Battle of the sexes: The best responses for player 1 are underlined

		2	
		O	M
1	O	$\underline{3}$, 1	0 , 0
	M	0 , 0	$\underline{1}$, 2

Example

Battle of the sexes: The best responses for player 2 are underlined

		2	
		O	M
1	O	3 , $\underline{1}$	0 , 0
	M	0 , 0	1 , $\underline{2}$

Example

The best responses for both players are underlined

$$
\begin{array}{ccc|c|c|}
 & & \multicolumn{2}{c}{2} \\
 & & \multicolumn{1}{c}{O} & \multicolumn{1}{c}{M} \\
\cline{3-4}
1 & O & \underline{3}\ ,\ \underline{1} & 0\ ,\ 0 \\
\cline{3-4}
 & M & 0\ ,\ 0 & \underline{1}\ ,\ \underline{2} \\
\cline{3-4}
\end{array}
$$

Therefore we have two pure NE: (O, O) and (M, M)

Coordination

- This example shows that to obtain a NE we need some sort of coordination in addition to common knowledge of the game and rationality, which were the assumptions we needed to obtain an equilibrium in dominance solvable games
- How can we predict which one is the most likely equilibrium?

- *Schelling's (1960) theory of focal points suggests that in some "real life" situations players may coordinate on a particular equilibrium by using information not considered in the strategic form
- *This "focalness" depends on players' culture and past experience
- For example, in the Battle of the sexes players may coordinate on going to the opera if the previous weekend they went to the movies
- Also the focal point when choosing between Left and Right when driving may vary across countries
- Pareto dominance can also be used as a mechanism for coordination, that is players could prefer an equilibrium where the payoff for each agent is strictly greater than the respective payoffs in a different equilibrium

Non-existence of pure NE

Pure strategy Nash equilibrium may not alway exist.

Example

Consider the following game called **Matching Pennies**. Two players simultaneously and independently select "heads" or "tails" by each uncovering a penny in his hand. If selections match, then player 2 must give his penny to player 1; otherwise player 1 gives his penny to player 2. Note that it is a zero-sum game

		2	
		H	T
1	H	$1\ ,\ -1$	$-1\ ,\ 1$
	T	$-1\ ,\ 1$	$1\ ,\ -1$

Example

This game does not have a NE in pure strategies since best responses do not match

$$
\begin{array}{ccc|c|c|}
 & & \multicolumn{2}{c}{2} \\
 & & \multicolumn{1}{c}{H} & \multicolumn{1}{c}{T} \\
\cline{3-4}
1 & H & \underline{1}\ ,\ -1 & -1\ ,\ \underline{1} \\
\cline{3-4}
 & T & -1\ ,\ \underline{1} & \underline{1}\ ,\ -1 \\
\cline{3-4}
\end{array}
$$

Mixed Strategy NE

If we take mixed strategy into consideration, NE always exists.

Definition

$(\sigma_1^*, ..., \sigma_N^*)$ is a Nash equilibrium in mixed strategies if

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma_i', \sigma_{-i}^*) \quad \text{for all } \sigma_i' \in \Delta(S_i)$$

where $U_i(\sigma_i, \sigma_{-i})$ is player i's *expected payoff.*

Definition

$(\sigma_1^*, ..., \sigma_N^*)$ is a Nash equilibrium if for each player i, the strategy σ_i^* of player i is a best response to σ_{-i}^*, i.e.

$$\sigma_i^* \in BR_i\left(\sigma_{-i}^*\right)$$

Some Properties of NE

- There will always exist a NE (in mixed strategies) for any game with finite strategies.
  - *Actions are now in an interval (between 0 and 1)
  - *Payoffs are continuous since they are linear in the actions (expected utility)
  - *These imply that best responses are defined on intervals and are continuous
- *For almost all games, there are an odd number of equilibria (which might be infinity)

How to solve for mixed strategy NE?

Example

- Suppose player 1 plays the mixed strategy

$$\sigma_1 = \begin{cases} H & \text{with probability } = p \\ T & \text{with probability } = (1-p) \end{cases}$$

- while player 2 plays the mixed strategy

$$\sigma_2 = \begin{cases} H & \text{with probability } = q \\ T & \text{with probability } = (1-q) \end{cases}$$

- $BR_1(\sigma_2)$: For player 1,

$$\begin{cases} U_1(H,\sigma_2) = q \cdot (1) + (1-q) \cdot (-1) = 2q - 1 \\ U_1(T,\sigma_2) = q \cdot (-1) + (1-q) \cdot (1) = 1 - 2q \end{cases}$$

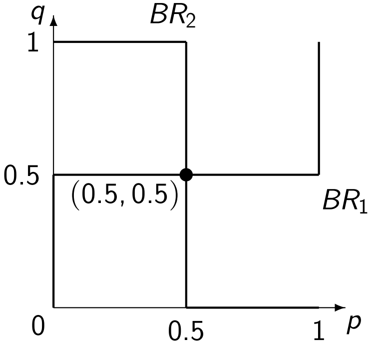
- If $1-2q > 2q-1$, i.e. $q < \frac{1}{2}$, then $BR_1 = \{T\}$. i.e. $p = 0$.
- If $1-2q < 2q-1$, i.e. $q > \frac{1}{2}$, then $BR_1 = \{H\}$. i.e. $p = 1$.
- If $1-2q = 2q-1$, i.e. $q = \frac{1}{2}$, then $BR_1 = \{H, T, \text{any } (p, 1-p) \text{ on } H \text{ and } T\}$. $p \in [0,1]$ since $U_1(H, \sigma_2) = U_1(T, \sigma_2) = 0$.

- $BR_2(\sigma_1)$: For player 2,

$$\begin{cases} U_2(H, \sigma_1) = p \cdot (-1) + (1-p) \cdot (1) = 1 - 2p \\ U_2(T, \sigma_1) = p \cdot (1) + (1-p) \cdot (-1) = 2p - 1 \end{cases}$$

- If $1-2p > 2p-1$, i.e. $p < \frac{1}{2}$, then $BR_2 = H$. i.e. $q = 1$.
- If $1-2p < 2p-1$, i.e. $p > \frac{1}{2}$, then $BR_2 = T$. i.e. $q = 0$.
- If $1-2p = 2p-1$, i.e. $p = \frac{1}{2}$, then $BR_1 = \{H, T, \text{any } (p, 1-p) \text{ on } H \text{ and } T\}$. $q \in [0,1]$ since $U_2(\sigma_1, H) = U_2(\sigma_1, T) = 0$.

- A graph shows the interaction of best responses



- Mixed NE: $\sigma_1^* = BR_1(\sigma_2^*)$ and $\sigma_2^* = BR_2(\sigma_1^*)$ yield $p = q = \frac{1}{2}$. i.e. player 1 plays the mixed strategy

$$\sigma_1^* = \begin{cases} H & \text{with probability } = 0.5 \\ T & \text{with probability } = 0.5 \end{cases}$$

while player 2 plays the mixed strategy

$$\sigma_2^* = \begin{cases} H & \text{with probability } = 0.5 \\ T & \text{with probability } = 0.5 \end{cases}$$

- In this equilibrium, all four outcomes of the game occurs with equal probabilities, and each player gets an expected payoff of zero.

Example

Modified Matching Pennies

		2: H	2: T
1	H	-3 , 1	1 , -1
	T	1 , -1	-1 , 1

- In this variation the NE is given by player 1,s mixed strategy

$$\sigma_1^* = \begin{cases} H & \text{with probability} & = \frac{1}{2} \\ T & \text{with probability} & = \frac{1}{2} \end{cases}$$

while player 2 plays the mixed strategy

$$\sigma_2^* = \begin{cases} H & \text{with probability} & = \frac{1}{3} \\ T & \text{with probability} & = \frac{2}{3} \end{cases}$$

- Therefore the strategy profiles are now played in equilibrium according to the following probability distribution

$$\begin{cases} \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} & \text{chance of} \quad (H, H) \\ \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} & \text{chance of} \quad (H, T) \\ \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} & \text{chance of} \quad (T, H) \\ \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} & \text{chance of} \quad (T, T) \end{cases}$$

- Note that while the modification was in the payoff for player 1, the equilibrium strategy changes for player 2
- This is so because player 2's strategy shapes the behavior of player 1 and plays the role of making him indifferent between her two strategies; thus it has to place a lower probability to the worst outcome
- In a mixed strategy Nash equilibrium, players are **indifferent** among pure strategies whose actions are assigned by positive probabilities.

- By definition, mixed strategy means that for each player i, its mixed strategy σ_i is a best response of other players' σ_{-i}.
- But when we solve for mixed strategy NE, we apply the fact that **players are indifferent among the actions that are assigned by strictly positive probabilities**.
- For instance, if A_i and B_i are two actions that are played with positive probability by player i in a mixed strategy Nash equiblirium, the expected payoffs generated by playing A_i and B_i must be equal to each other.
- *Why? To informally prove this, we can assume the opposite, say playing A_i generates strictly larger expected payoff than playing B_i, then a strategy in which probability of playing B_i is zero can generate strict larger payoff. Then the original payoff is not a best response. Contradictary to the definition of NE!

Example

Consider the following game

		2	
		C	D
1	A	4 , 7	1 , 3
	B	2 , 0	9 , 5

- But double-underline algorithm, it is easy to find pure strategy Nash Equilibria of the game (A, C) and (B, D).

Example

To solve for mixed strategy Nash Equilibrium

$$
\begin{array}{ccc|c|c|}
 & & & \multicolumn{2}{c}{2} \\
 & & & \multicolumn{1}{c}{\underset{q}{C}} & \multicolumn{1}{c}{\underset{1-q}{D}} \\
\cline{4-5}
1 & & \underset{p}{A} & 4\ ,\ 7 & 1\ ,\ 3 \\
\cline{4-5}
 & & \underset{1-p}{B} & 2\ ,\ 0 & 9\ ,\ 5 \\
\cline{4-5}
\end{array}
$$

- We let p be player 1's probability of playing A; and q is player 2's probability to play C.

Example

To make $(p, 1-p)$ and $(q, 1-q)$ strictly positive, we apply the property of indifference.

- Given player 2 playing $(q, 1-q)$, player 1 is indifferent between playing A and B

$$\begin{aligned} 4 \cdot q + 1 \cdot (1-q) &= 2 \cdot q + 9 \cdot (1-q) \\ q &= \frac{4}{5} \end{aligned}$$

- Given player 1 playing $(q, 1-q)$, player 2 is indifferent between playing C and D

$$\begin{aligned} 7 \cdot p + 0 \cdot (1-p) &= 3 \cdot p + 5 \cdot (1-p) \\ p &= \frac{5}{9} \end{aligned}$$

A comprehensive example

- Now we have discussed sevaral tools to solve for a game
  - Deleting strict domintated strategies or finding strict dominant strategy
  - Double underline algorithm
  - Using indifferences to find mixed strategy NE
- Now let's consider a game which requires multiple tools to solve.

- Consider the 4 entrepreneurs who want to start business in an industry
- Players: 4 firms A, B, C, D
- Actions/strategies: $S_i = \{Entry, No\}$
- Payoffs:
  - $u_i\left(No, s_{-i}\right) = 0$
  - $u_i\left(Entry, s_{-i}\right) = \pi\left(N\right) - c_i$ where
    - Costs: $c_A = 100, c_B = 300, c_C = 300, c_D = 500$
    - Profits per entrant as a function of number of entrants: $\pi\left(1\right) = 1000, \pi\left(2\right) = 400, \pi\left(3\right) = 250, \pi\left(4\right) = 150$

- The first thing to observe is that *Entry* is strictly dominant for firm A, because $c_A = 100 < \pi(n)$ for $n = 1, 2, 3, 4$.
- Second, if firm A enters, *Entry* is strictly dominated for firm D, because $c_D = 500 > \pi(n)$ for $n = 2, 3, 4$.
- So, we know that in equilibrium, A will enter and D will not, and can focus on firms C and B.

- The matrix of the reduced game is given by

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{\text{C}} \\
 & & E & N \\
\cline{3-4}
\text{B} & E & -50\ ,\ -50 & 100\ ,\ 0 \\
\cline{3-4}
 & N & 0\ ,\ 100 & 0\ ,\ 0 \\
\cline{3-4}
\end{array}
$$

- This game has two pure strategy Nash equilibria, (N, E) and (E, N)

- There is also a mixed strategy equilibrium
- We let p be C's probability of playing E; and q is C's probability to play E. So we have

$$\begin{aligned} -50q + 100(1-q) &= 0 \Rightarrow q = 2/3 \\ -50p + 100(1-p) &= 0 \Rightarrow p = 2/3 \end{aligned}$$

- By finding strict dominant and dominated strategies, we reduce the game into a 2×2 game, $(p^* = 2/3, q^* = 2/3)$ is a mixed strategy Nash equilibrium.
- To summarize, the entry game has three Nash equilibria where firm A enters, and firm D stays out:
  1. Firm B enters and firm C stays out
  2. Firm C enters and firm B stays out
  3. Both B and C enter with probability $2/3$.

Interpretation of mixed strategy NE

- People really randomize. Examples for this are service in tennis, mixing in Rock Scissor Paper, penalty kicks.
- If players play the game several times with different players (or without remembering previous opponents), mixed strategies can be interpreted as the fraction of times that a strategy has been played
- Think on large populations where a player changes his or her opponent every period (matching games).
  - Probability of action a = Ratio of population who plays a.
  - A player's action in equilibrium is a best response to the distribution of actions in the population.

Nash Equilibrium and Pareto Optimum

Definition

A payoff vector $u = (u_1, u_2 ... u_n)$ Pareto dominates another vector $u' = (u'_1, u'_2 ... u'_n)$ if for all $i = 1, 2, 3 ... n$,

$$u_i \geq u'_i$$

and for at least some $j \in \{1, 2, 3 ... n\}$. such that

$$u_j > u'_j$$

Definition

A payoff vector u is Pareto optimal if there is no other vector that Pareto dominates u.

- In classical economics, Pareto dominance and Pareto optimum are used to measure social walfare.
- Nash equilibrium and Pareto dominance are not indicating each other.

Example

Prisoners' Dilemma

		2	
		C	D
1	C	$-1\ ,\ -1$	$-5\ ,\ 0$
	D	$0\ ,\ -5$	$-3\ ,\ -3$

(C, C) is Pareto optimal but (D, D) is the unique Nash equilibrium.

Example

Battle of Sex

		Boy: O	Boy: M
Girl	O	$3\ ,\ 1$	$-5\ ,\ 0$
	M	$0\ ,\ -5$	$1\ ,\ 2$

(O, O) and (M, M) are Pareto optimal and both are Nash equilibria.
There is a mixed strategy Nash equilibrium, $p = 7/8$, $q = 2/3$ (why?). But the expected payoffs are $\left(\frac{1}{3}, \frac{1}{4}\right)$ which is Pareto dominated by both (M, M) or (O, O).

Example

For zero-sum games, all outcomes are Pareto optimal because total payoffs are always constant.

Weak dominance and Nash Equilibrium

Recall what is weak dominance

Definition

Strategy s_i weakly dominates s_i' if for EVERY strategy profile of the others $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$, i weakly prefers s_i over s_i'

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

Definition

s_i strictly dominates ALL other strategies of player i, then we say s_i is **the weakly dominant** strategy for player i.

Definition

A pure strategy s_i is weakly dominated if there is a (mixed) strategy σ_i such that $u_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for all strategy profiles s_{-i}

- Avoiding weakly dominated strategy may help us to find NE. But we may miss some NE.
- If we iteratively eliminate weakly dominated strategies and are left with a unique strategy profile, then it is a NE
- But we cannot assure this would be the unique NE
- We can use this method to find a Nash Equilibrium but we cannot find all of them.

Example

Consider the following game

		Boy: Date	Boy: TV
Girl	Date	$(1,1)$	$(0,0)$
	TV	$(0,0)$	$(0,0)$

- "*Date*" is a weakly dominant strategy for both players.
- But by double underline algorithm, we can see that both $(Date, Date)$ and (TV, TV) are pure strategy NE
- If both players believe that the other stays home watching TV, (TV, TV) is a valid NE.

Second Price Auction

- Second price auction is a commonly used auction mechanism.
- Many auction platforms is using such mechanism, e.g. eBay.
- Players submit their bids for an item.
- The auction stops when no one submit higher bids. The buyer with the highest bid wins the item.
- But the winner only pays the money equal to second highest bid
- *If the winner pays his own bid, the auction is called first price action.

- Formally, the game is as follows
- Players: N bidders: $I = \{1, 2, 3, ...N\}$
    - Each bidder has a private valuation on the item v_i
- Actions: $b_i = \{\$0, \$1, \$2, \$3, \$4...\}$
    - Each bidder can only choose integer amount of money
    - Each bidder can only submit one bid simultaneously.
- Payoffs:

$$U_i(b_i) = \begin{cases} v_i - b^{\text{second}} \text{ if } b_i = \max\{b_1, b_2, b_3 ... b_N\} \\ 0 \text{ otherwise} \end{cases}$$

- The game is not easy to solve by any tools we have discussed before.
- Fortunately, we can prove that $b_i = v_i$ is player's weakly dominant strategy. If so, bidding their values is a NE equilibrium.

- The following table summerizes bidder $i's$ payoffs comparing bidding v_i and another $b_i \neq v_i$. Denote b^* is the highest bid of other players. If bidder i wins, it has has to pay his bid minus b^*
- If b_i is smaller than v_i

		Bidding v_i		Bidding b_i	
		Result	Utility	Result	Utility
1	$b_i < v_i < b^*$	Lose	0	Lose	0
2	$b_i < b^* < v_i$	Win	$v_i - b^* > 0$	Lose	0
3	$b^* < b_i < v_i$	Win	$v_i - b^* > 0$	Win	$v_i - b^* > 0$

- If b_i is larger than v_i

		Bidding v_i		Bidding b_i	
		Result	Utility	Result	Utility
1	$v_i < b_i < b^*$	Lose	0	Lose	0
2	$v_i < b^* < b_i$	Lose	0	Win	$v_i - b^* < 0$
3	$b^* < v_i < b_i$	Win	$v_i - b^* > 0$	Win	$v_i - b^* > 0$

- We can see that, bidding v_i always weakly dominates other bids, no matther what other buyers bid.
- Although there may be more than one NE in this game, weak dominance at least helps us find one.

Why should we consider Nash equilibrium?

- **Self-enforcing**
  - Once players agree to play a NE, then they have incentive to play the NE (as long as all the other players do) without any external enforcement.
  - This implies that NE are consistent predictions of how the game will be played, since it has the property that the players can predict it, predict that their opponents predict it, and so on

- **Learning**
  - Suppose that players play the same game over and over, and after a while, they settle on some strategy s^* and play it period after period.
  - Suppose also that they only care about their current payoffs when choosing their strategies.
  - Since they should expect that s^* is played in the next period in such a situation, there should not exist any profitable unilateral deviation from s^*.
  - This implies that s^* must be a NE. This suggests that the stable outcome which is reached over time must be a NE.

- **Evolution and Adaptation**
  - If players always adapt to a better strategy (maybe by experimentation or evolution), the stable outcome must be a Nash equilibrium by the same reason as above.
  - So players do not necessarily have to be rational to play Nash equilibrium. Even some insects and plants play NE!

*Evolutionary game theory

- In theoretical biology, Maynard Smith and Price (1973) pioneered the idea that animals are genetically programmed to play different pure strategies
- The genes whose strategies are more successful are the ones who have higher reproductive fitness
- So players do not necessarily have to be rational to play Nash equilibrium. Even some insects and plants play NE!

- Why is there a roughly even number of men and women in the world?
- The strategies for the genes consist on giving birth a women or a man
- The payoffs are given by the number of offsprings

- Assume that the initial genes play a strategy where the ratio of women to men $\frac{w}{m}$ was not equal to 1 but, say, $\frac{w}{m} = 0.2$, so there are five men for each woman
- Suppose every woman has $N = 12$ kids, 2 daughters and 10 sons

- Every man has a chance of $\frac{1}{5}$ of mating with a woman and thus in expectation has 0.4 daughters and 2 sons
- Note that women are much more "productive" than men since each women generates more offsprings than a man
- So, every woman will in expectation have 8 granddaughters and 40 grandsons:
  - She has 12 children, 2 daughters and 10 sons
  - Each of the 2 daughters has 2 daughters and 10 sons for a total of 4 granddaughters and 20 grandsons
  - Each of the 10 sons has 0.4 daughters and 2 sons for a total of 4 granddaughters and 20 grandsons

- Consider now a mutation whose strategy leads to an equal number of 6 girls and 6 boys
    - She still has 12 children, but this time they are 6 daughters and 6 sons
    - Each of the 6 daughters has 6 daughters and 6 sons
    - Each of the 6 sons has 1.2 daughters and 1.2 sons (remember the mutation occurs in the previous population where the ratio was 0.2)
    - Thus, the total number of grandchildren is $6 * 6 + 6 * 1.2 = 36 + 7.2 = 43.2$ granddaughters and 43.2 grandsons
    - ...

- Thus, after only two generations the mutation is leading to strictly more granddaughters and grandsons
- Therefore the mutation will eventually spread over the entire population
- Hence the initial mixed strategy (genes) was not a NE since there was another mixed strategy (mutation) that generated greater payoffs (offsprings)

Game Theory 106G
Lecture 4: Game with continous actions

Menghan Xu

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Continuum of actions

- We will now study applications of the pure NE in games with a continuum of actions
- Now actions belong to intervals of real numbers instead of discrete sets.
- *We will usually impose continuity and concavity to the payoffs to assure the existence of the equilibrium
- *Weaker conditions can also assure the existence of equilibria, but this is enough for our purpose.
- We will not focus on mixed strategies in this environment, though we could have such generalization

Cournot competition

- Cournot competition is also called quantity competition.
- Firms produce goods of the same quality. Their total production will determine market price.
- Each firm's chosen quantity has impact on price. But no one has monopoly power.

Cournot competition

- Players: n firms - $i = 1, ..., n$
- Action set: Each firm i decides to produce $q_i \in A_i = [0, \infty)$ of a good
- Payoffs: $u_i(q_1, ..., q_n) = p(Q)\, q_i - c_i(q_i)$ where $Q = \sum_{j=1}^{n} q_j$ is the total amount of the good produced
- $p(Q)$ is called inversed demand function. In general, it is a downward sloping curve.

- To be concrete, we assume
  - linear demand: $p = 1000 - Q$
  - 2 firms: $i = 1, 2$
  - Linear and identical costs: $c_i(q_i) = 100q_i$

- Remember a NE is defined as the point where all best responses intersect, so we need to find (q_1^*, q_2^*) such that q_1^* is a best response to q_2^* and vice versa
- Given quantity chosen by the competitor q_2, firm 1 maximizes his profit:
- Remember profit is defined by total revenue minus total cost. So firm 1 solves the following problem:

$$\begin{aligned}\max_{q_1} u_1(q_1, q_2) &= p(Q) \cdot q_1 - c_1(q_1) \\ &= (1000 - q_1 - q_2) \cdot q_1 - 100 q_1\end{aligned}$$

- Notice that the profit is a function of q_2 since it affects market price
- *Note also that profit is a concave function of q_1 and therefore the first order condition (FOC) is necessary and sufficient to characterize the optimum
- The FOC, which can be stated as Marginal Revenue equal to Marginal Cost, yields

$$\begin{aligned} p + \partial p / \partial q_1 \cdot q_1 &= 100 \\ 1000 - 2q_1 - q_2 &= 100 \\ q_1 &= \frac{900 - q_2}{2} \end{aligned}$$

- Hence, the best response $q_1 = BR_1(q_2)$ to his beliefs over q_2 is

$$q_1 = BR_1(q_2) = \frac{900 - q_2}{2}.$$

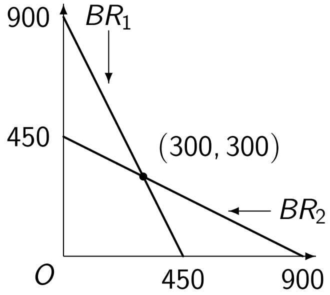
- Similarly the best response for firm 2 given q_1 is

$$BR_2(q_1) = \frac{900 - q_1}{2}.$$

- A NE is then a pair (q_1^*, q_2^*) such that $q_1^* = BR_1(q_2^*)$ and $q_2^* = BR_2(q_1^*)$.
- Thus, we have to solve the system of equations

$$q_1 = \frac{900 - q_2}{2}$$
$$q_2 = \frac{900 - q_1}{2}$$

- A graph shows the intersection of best responses



- The solution to the equations yields $q_1^* = q_2^* = 300$, thus the market price $p = 400$, and payoffs are given by

$$\begin{aligned} u_1(300, 300) &= 400 \cdot 300 - 100 \cdot 300 = 90,000 \\ u_2(300, 300) &= 400 \cdot 300 - 100 \cdot 300 = 90,000 \end{aligned}$$

Cournot competition is dominance solvable

- Is it possible to deduce the equilibrium by iterated deletion of strictly dominated strategies? Or rationlization by interating over best responses?
- Consider again firm 1's best response function

$$BR_1(q_2) = 450 - q_2/2$$

- Starting from $BR_1^0 = BR_2^0 = [0, \infty)$.
- The smallest quantity that can be chosen by firm 2 is $q_2 = 0$, hence firm 1 will never choose a quantity above 450
- Since firm 1 is also constrained to set positive quantities then we can say that $q_1^* \in BR_1^1 = [0, 450]$ for any quantity $q_2 \geq 0$
- Alternatively, we can argue that any strategy $q_1 > 450$ is strictly dominated by $q_1' = 450$, because u_1 is decreasing in q_1 for any (q_1, q_2) with $q_1 \geq 450$ since

$$\frac{\partial}{\partial q_1} u_1(q_1, q_2) = 900 - 2q_1 - q_2 \qquad (*)$$

- Therefore firms can conclude that $q_1 \in [0, 450]$ and also that $q_2 \in [0, 450]$ by the same argument
- What can firm 1 learn from this step?

- If firm 1 knows that $q_2 \in [0, 450]$, then his best responses are in the interval $BR_1^2 = [450 - 450/2, 450 - 0] = [225, 450]$
- Alternatively we can argue that u_1 is strictly increasing in q_1 for any (q_1, q_2) with $q_1 \leq 225$ and $q_2 \leq 450$, so any $q_1 \leq 225$ is strictly dominated by $q_1' = 225$ in the game with strategy sets $BR_i^1 = [0, 450]$

- Therefore firms can conclude that $q_1 \in [225, 450]$ and symmetrically that $q_2 \in [225, 450]$
- Again, what can firm 1 learn from this in turn?

- If firm 1 knows that $q_2 \geq 225$, then his best responses are in the interval
$BR_1^3 = [450 - 450/2, 450 - 225/2] = [225, 337.5]$
- Alternatively we can argue that u_1 is strictly decreasing in q_1 for any (q_1, q_2) with $q_1 \geq 337.5$ and $q_2 \in [225, 450]$, so any $q_1 \geq 337.5$ is strictly dominated by $q_1' = 337.5$ in the game with strategy sets $BR_i^2 = [225, 450]$
- ...

- If we iterate this process infinitely many times, the only strategy left for each firm is $q_i = 300$.
- However, the success of iterated deletion of strictly dominated strategies in the Cournot setting hinges on the assumption that there are only two firms.
- *For three or more symmetrical firms the process terminates after deleting the dominated strategies in the first step (why?)

Collusive Outcome - Cartel

- Suppose the firms form a cartel and jointly set q_1 and q_2. Then they maximize their total utility, in other words, they solve the monopoly maximization problem

$$\max_{Q}(1000 - Q) \cdot Q - 100Q = u_1 \left(q_1, q_2\right) + u_2 \left(q_2, q_1\right)$$

- The first-order condition (marginal revenue = marginal cost) yields

$$\begin{aligned} p + \partial p/\partial Q \cdot Q &= 100 \qquad (**) \\ 1000 - 2Q &= 100 \end{aligned}$$

- Compare this first order condition of the cartel (**) to the first order condition of an individual firm (*).
- The Marginal revenue in the cartel case considers the effect of increasing the quantity in both payoffs and therefore the decrease in the price $\partial p/\partial Q$ is more heavily weighted
- As the cartel suffers more from producing the extra marginal quantity than the individual firm does, the resulting equilibrium quantity of the cartel should be expected to be lower than the equilibrium quantity in the duopoly.

- Indeed we find that $Q^* = 450$, which implies a market price of 550, and a joint profit of $202,500$; which is greater than the total profit found in the NE $(180,000)$
- This means that the firms can do better if they reduce competition and collude.
- But do they have the incentive to collude?

- Suppose one of the firms wants to cooperate and produces $q_1^* = 225 = 450/2$, and of course expects firm 2 to produce exactly the same in order to maximize the joint profits, which will yield each of them $101, 225$
- But firm 2 does not think in the same way. According to her best response ($BR_2(q_1) = \frac{900 - q_1}{2}$), she will produce $q_2 = 337.5$

- This strategy profile implies a market price of 437.5, which will yield firm 2 an utility of 113, 910 and firm 1 a profit of 75, 937.5, much lower to what she expected
- If we consider a simplified 2×2 game

		2: C	2: D
1	C	101, 225 , 101, 225	75, 937.5 , 113, 910
	D	113, 910 , 75, 937.5	90, 000 , 90, 000

- It is a prisoner dilemma situation.

Fully Competitive Outcome

- Another useful reference point is the fully competitive equilibrium in which many small firms take prices as given.
- In this case the profits have to be zero, otherwise more firms will enter the market and lower the price until profits are zero
- As we assumed constant marginal costs of 100 the equilibrium price must be 100 as well, yielding an equilibrium quantity of $Q = 900$.
- Note this case, there are too many firms and any single firm cannot affect the price.

- To summarize:

	# of firms N	Total Quantity	Price	Total Profit
Cartel / Monopoly:	1	450	550	$202,500$
Duopoly	2	600	300	$180,000$
Oligopoly	$3,4,5...$	?	?	?
Competitive	∞	900	100	0

Bertrand competition

- Bertrand competition is also called price competition
- In many markets it is more plausible to think of firms setting prices, while quantities are determined by consumer demand; for example, Shell and Chevron across a street setting their prices.
- Suppose products are completely homogeneous, such that consumers choose solely based on price.
- Also assume that neither firm is capacity-constrained - they can produce as much as possible.

- Players: $N = 2$ firms
- Strategy set: $S_1 = S_2 = [0, \infty)$. Firm 1 chooses its price $p_1 \in S_1$, and firm 2 chooses its price $p_2 \in S_2$.
- The (constant) marginal cost is $c_1 = c_2 = 100$ and demand is given by

$$Q(p) = 1000 - \underline{p}$$

where $\underline{p}$ is the lower price.

- Assume all consumers choose the firm with strictly lower price. If the prices are the same, the two firms equally split market demand.
- Payoffs are given by

$$\pi_1(p_1, p_2) = \begin{cases} (p_1 - c_1) \cdot (1000 - p_1) & \text{if } p_1 < p_2 \\ (p_1 - c_1) \cdot \frac{1000 - p_1}{2} & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2 \end{cases}$$

$$\pi_2(p_1, p_2) = \begin{cases} (p_2 - c_2) \cdot (1000 - p_2) & \text{if } p_2 < p_1 \\ (p_2 - c_2) \cdot \frac{1000 - p_2}{2} & \text{if } p_2 = p_1 \\ 0 & \text{if } p_2 > p_1 \end{cases}$$

Nash equilibrium

- Since the monopoly optimal price is 550 (why?), the firm 1's best response is

$$BR_1(p_2) = \begin{cases} 550 & \text{if } p_2 > 550 \\ p_2 - \varepsilon & \text{if } 100 < p_2 \leq 550 \\ \text{any } p_1 \geq p_2 & \text{if } p_2 = 100 \\ \text{any } p_1 > p_2 & \text{if } p_2 < 100 \end{cases}$$

where ε denotes a very small positive number (e.g. 0.0001). Here it denotes a slight undercut of the price

- By symmetry, the best response for firm 2 is defined in the same way
- (p_1^*, p_2^*) is a NE if and only if $p_1^* = BR_1(p_2^*)$ and $p_2^* = BR_2(p_1^*)$.
- It is easy to see that pricing at $p_1^* = p_2^* = 100$ is an equilibrium, which yield zero profits.

- Actually it is the only equilibrium because:
  - (p_1, p_2) with either $p_1, p_2 < 100$ cannot be an equilibrium: The firm with the lower price is making losses and could do better by pricing above 100
  - (p_1, p_2) with both $p_1, p_2 > 100$ cannot be an equilibrium: The firm with the higher price could do better by slightly undercutting its competitor
  - (p_1, p_2) with $p_1 > p_2 = 100$ (or vice versa) cannot be an equilibrium: Firm 2 could do better by pricing more closely to its competitor

Zero profits

- One troublesome consequence of the above analysis is that firms do not make any profits in this game.
- But if firms make zero profits why would they ever invest to enter the market in the first place?
- Profits can be restored when
  - Products are differentiated
  - Firms are capacity constrained
  - Firms have different costs

Capacity constraints

*

- Assume that every firm can produce at most 400 units of the good.
- For simplicity we now set cost equal to 0.
- Then payoffs become

$$u_1(p_1, p_2) = \begin{cases} p_1(1000 - p_1) \text{ but no more than } 400p_1 & \text{if } p_1 < p_2 \\ p_1 \frac{1000-p_1}{2} \text{ but no more than } 400p_1 & \text{if } p_1 = p_2 \\ p_1\,(600 - p_1) \text{ but no more than } 400p_1 & \text{if } p_1 > p_2 \end{cases}$$

- In the first case, firm 1 is offering the better price and can satisfy all demand $1000 - p_1$ up to its capacity of 400.
- In the second case, the firms are offering the same prices and they share the market $\frac{1000-p_1}{2}$ up to their respective capacities of 400.
- In the third case, firm 2 is offering the better price and it gets to provide 400 units to the market so that the remaining demand for the products of firm 1 is $Q(p_1) = 600 - p_1$.
- In the practice problems you are asked to show that there is no Nash equilibrium (in pure strategies).

Different Unit costs

- If firm 2's cost are 200 instead of 100, its best response function becomes

$$BR_2(p_1) = \begin{cases} 600 & \text{if } p_1 > 600 \\ p_1 - \varepsilon & \text{if } 200 < p_1 \leq 600 \\ \text{any } p_2 \geq p_1 & \text{if } p_1 = 200 \\ \text{any } p_2 > p_1 & \text{if } p_1 < 200 \end{cases}$$

- Now, for any $p \in [100, 200]$ there is a Nash equilibrium, where firm 2 chooses $p_2 = p$ and firm 1 chooses $p_1 = p - \varepsilon$.
- Firm 2 has no incentive to undercut firm 1, because this would mean pricing below its costs of 200.
- Firm 1 on the other hand is making a profit by undercutting firm 2 as long as $p_1 \geq 100$, and the most profitable way of doing so is by undercutting p_2 by as little as possible, i.e. $\varepsilon \approx 0$.
- The most reasonable Nash equilibrium is $(p_1^*, p_2^*) = (200 - \varepsilon, 200)$, because all other equilibrium strategies $p_2 < 200$ of firm 2 are weakly dominated by $p_2^* \leq 200$.

Game Theory 106G
Sequential Game

Menghan Xu

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Dynamic Games with Perfect Information

- In many situations, players move sequentially rather than simultaneously, for example,
    - real games: chess, checkers
    - in sports: golf, snooker, etc
    - in business: patent races, incumbent firms vs new entrants, workers vs managers, etc
    - in politics: politicians (campaign) vs voters, terrorists vs CIA, etc.
    - in macroeconomics: "time consistency" of policies
- We will first focus on games with **perfect information** where players can observe previous actions taken by other players, as well as every relevant information of the game

- To model this dynamic situations we use games in extensive form
- Remember the previous games represented by payoff matrix are called strategic or normal form
- An extensive form specifies the order of moves, actions can be taken at each move and final payoffs of each history. It can be viewed as a decision tree

Extensive Form

Definition

The extensive form of a dynamic game with perfect information consists of

- A set of *players:* $I = \{1, 2, 3, \cdots, n\}$.
- The order of move represented by a *game tree - nodes and paths*
  - **Paths:** Each path represents an action.
  - **Action nodes:** Each action node specifies whose turn to move. It extends to several actions that could be played by the player. It is reached by a single path (action taken by previous player);
  - **Terminal nodes:** Each terminal node is reached by a action taken by last mover. And it specifies payoffs of each player.

Example

Entry into an industry: There is an incumbent in a industry. There is a potential entrant who considers whether to enter the industry.

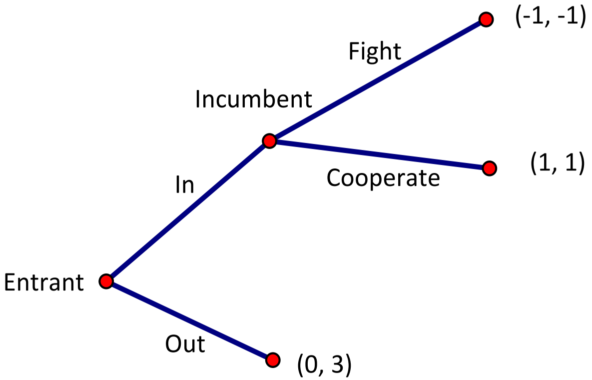
Example

In this example we have

- Two players (Entrant and Incumbent),
- Two action nodes that specify that Entrant plays first and Incumbent moves next.
- Two actions at each action node,
- and three terminal nodes with their respective payoffs

Example

Game tree



Strategies and NE in Sequential Games

Definition

A **strategy** s_i of player i is a complete **contingent plan**, i.e. a strategy s_i needs to specify actions to take at each action node of player i.

- When we studied one-period game, the terms "action" and "strategy" can be used interchangeably. But in sequential game, strategy is a "combination" of actions.

- In the example, strategies are the same as actions since the Incumbent only gets to play if the Entrant enters. In other words, the each mover has only one action node. But in more complicated game, it will not be the case.
- We can also represent this game as a normal form and find Nash equilibria.

		Incumbent	
		Fight	Cooperate
Entrant	In	$(-1, -1)$	$(\underline{1}, \underline{1})$
	Out	$(\underline{0}, \underline{3})$	$(0, \underline{3})$

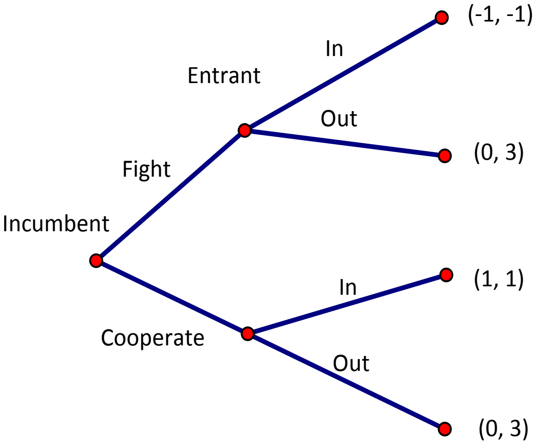
- There are two pure NE: (Out, Fight) and (In, Cooperate)
- By definition, nobody has to deviate in the normal representation
- How to under stand the NE (Out, Fight)?
  - Incumbent threats entrant: "If you enter the market, I will start a price war!"

Credibility of NE

- Is such threat credible that the Incumbent will Fight if the Entrant decides to Enter?
- No, because after entrant chooses "In", "Cooperate" dominates "Fight" for incumbent firm;
- Since the entrant moves first, he could force the incumbent to play the NE he wants: (In, Cooperate).
- Thus, the NE (Out, Fight) relies on an "empty threat"

Example

If Incumbent moves first, the game tree becomes



- In this game, Incumbent has one action node. Entrant has two action nodes.
- Incumbent's strategy set:

$$\{\text{Fight, Cooperate}\}$$

- Entrant's Strategy Set:

$$\{(\text{In, In}), (\text{In, Out}), (\text{Out, In}), (\text{Out, Out})\}$$

where (Out, In) means that entrant chooses "out" at first action node and chooses "In" at second action node.

- Alternatively, you can write (Out, In) in "conditional form", which is clear

$$(\text{Out}|\text{Fight, In}|\text{Cooperate})$$

- Since entrant has 4 strategies and incumbent has 2, if we want to translate the game tree into normal form, we need a 4×2 matrix

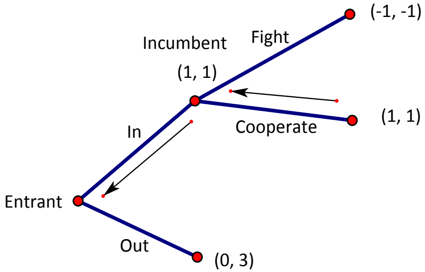
		Incumbent	
		Fight	Cooperate
Entrant	(In, In)	$(-1, -1)$	$(\underline{1}, \underline{1})$
	(In, Out)	$(-1, -1)$	$(0, \underline{3})$
	(Out, In)	$(\underline{0}, \underline{3})$	$(\underline{1}, 1)$
	(Out, Out)	$(\underline{0}, \underline{3})$	$(0, \underline{3})$

- We have three pure NE. But some are unlikely to happen when we take timing into consideration.

Backward Induction

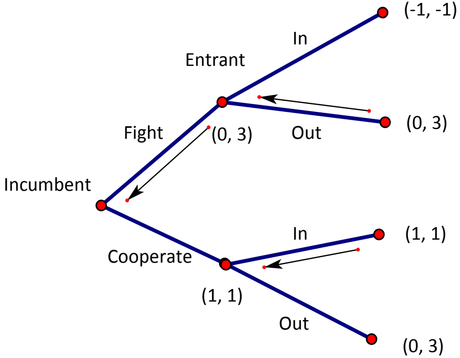
- In order to eliminate unreasonable NE and keep the better ones, let's introduce an intuitive procedure (or refinement) called "backward induction". Here is how to do it:
  1. At every last decision node (the nodes connected to terminal nodes), determine the optimal action for the player at the turn (if there are more than one optimal actions, keep all of them). Highlight the action path.
  2. Move the corresponding payoff vector to this action node.
  3. The action path will point back to a node by previous mover's action node, determine the optimal action, assuming that the actions determined in step 1 will be taken at the last node
  4. Repeat this procedure until the initial node is reached

Example



- This procedure yields an strategy profile which is also a NE
- Apply this concept to the entry game - at the last stage the incumbent will choose to Cooperate; knowing this, the entrant will choose to enter, which yields the NE (In, Cooperate)

Example



- Each player's actions are optimal at every possible history (yes, even at histories "off the equilibrium path")

Subgame

- So far we have two methods to find Nash equilibria in a sequential game. 1. Translate the extensive form into normal form; 2. Backward induction. But backward induction rules out some NE found by first method.

Definition

A subgame of a sequential game with perfect information is a subset of the game, such that

- It has a single initial action node
- It contains all the nodes(action nodes and terminal nodes) and paths that are successors of the initial node.
- It contains all the nodes that are successors of any node it contains.

A sequential game a subgame of itself.

Subgame Perfect Nash Equilibrium

- We can refine NE in sequential games using the concept of subgame

Definition

A subgame perfect Nash equilibrium(SPNE) is an strategy profile which represents a Nash equilibrium of every subgame of the original game.

- Equilibria found by backward induction are subgame perfect Nash equilibria.

Stackelberg Competition

Example

Stackelberg Competition: Two firms compete by choosing quantity. But firms take action sequentially.

- Company A chooses its quantity first. Knowing A's choice, company B chooses its quantity.
- For simplicity assume firms can chooses either high quantity H $(q_i = 5)$ or low L quantity $(q_i = 2)$
- Assume also that costs are zero, and that inversed demand is given by $p(Q) = 10 - Q$.

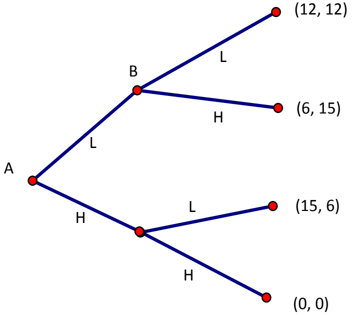
Example

Payoffs

- If both firms choose L, market price is 6 and each firm generates profit of 12.
- If one chooes L and the other chooses H, market price is 3. The firm chooses H earns 15 and the other earns 6.
- If both choose H, price becomes 0 and both earn zero.

Example

Game Tree



- There are totally 3 subgames.
- Transform the game to Normal form and find all Nash Equilibria

Example

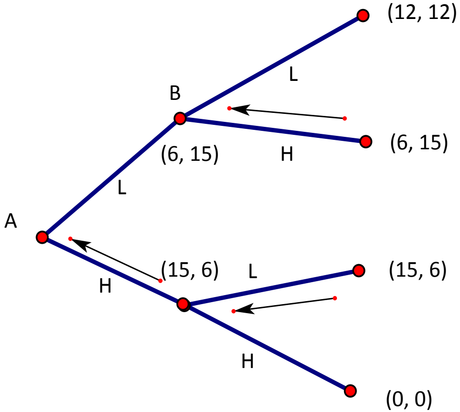
Normal form

	HH	HL	LH	LL
H	$0, 0$	$0, 0$	$15, 6$	$15, 6$
L	$6, 15$	$12, 12$	$6, 15$	$12, 12$

- There are three (pure strategy) Nash equilibria: $\{H, (L, H)\}$, $\{H, (L, L)\}$ and $\{L, (H, H)\}$

Example

Using backward induction



Example

Conclusion

- Unique SPNE is
$$\{H, (H, L)\}$$
- The SPNE of the game is that firm A chooses H; Firm B chooses H if A chooses L and chooses L if A choose H.
- **Important:** Although A will never play L in SPNE, we have to include the "off-equilibrium path" situation. Otherwise, it is not a complete SPNE

- If the extensive form is finite, so we can alway turn it into normal form, thus we know a NE must exist (Maybe mixed NE).

Theorem

(Zermelo's Theorem) Every **finite** *perfect information game has a backward induction solution(SPNE). Moreover, if no two payoffs are the same for any player, then there is a unique backward induction solution.*

- The theorem assures the existence of SPNE, which is one of the pure strategy NE.
- The assumption "finite" is important for the theorem. Finite means 1) each player can choose finite many actions at each node; 2) There are finite number of nodes.

Stackelberg Competition with a Continuum of Actions

- Consider the model studied for the Cournot competition
- There are 2 firms: $i = 1, 2$, and actions: $A_i = \{q_i \in [0; \infty)\}$
- There is a linear demand $p = 1000 - q_1 - q_2$, and linear costs $100q_i$
- The profits are

$$\begin{aligned} \Pi_1(q_1, q_2) &= (1000 - q_1 - q_2) \cdot q_1 - 100q_1 \\ \Pi_2(q_1, q_2) &= (1000 - q_1 - q_2) \cdot q_2 - 100q_2 \end{aligned}$$

- Suppose firm 1 moves first
- Now strategies are given by

1. $S_1 = A_1 = \{q_1 \in [0; \infty)\}$
2. $S_2 = \{q_2 : A_1 \to [0; \infty)\}$, i.e a strategy is a (contingent) plan that tells firm 2 how much $q_2(q_1)$ to produce for every quantity q_1 that firm 1 may have produced

- Remember that now we have a continuum of branches associated with each possible quantity
- Can you predict who will generate more profits? Or equal?

- In the last node firm 2 has to maximize her profits given the quantity q_1 produced by firm 1, which she already knows since she is the last mover)

$$\begin{aligned}\max_{q_2} \Pi_2 (q_1, q_2) &= (1000 - q_1 - q_2) \cdot q_2 - 100 q_2 \\ FOC &: \quad q_2 = 450 - \frac{q_1}{2}\end{aligned}$$

- Since firm 2 can observe the quantity perfectly, the best response is no longer based on belief

- Now, we go one step back, Firm 1 will maximize its own profit Π_1 choosing q_1, knowing that firm 2's choice would be $q_2 = q_2(q_1) = 450 - \frac{q_1}{2}$:

$$
\begin{aligned}
\max \Pi_1 (q_1, q_2) &= (1000 - q_1 - q_2) \cdot q_1 - 100 q_1 \\
&= 900 q_1 - q_1^2 - \left(450 - \frac{q_1}{2}\right) q_1 \\
&= 450 q_1 - \frac{1}{2} q_1^2 \\
FOC &: 450 - q_1 = 0
\end{aligned}
$$

- Hence, the equilibrium strategies will be

$$\begin{aligned} q_1^* &= 450 \\ q_2^* &= 450 - \frac{q_1^*}{2} \end{aligned}$$

- **Important:** You cannot simply say that the equilibrium strategy profile is

$$q_1^* = 450; q_2^* = 225$$

- The equilibrium payoffs are

$$\begin{aligned} \Pi_1^* &= 101,250 \\ \Pi_2^* &= 50,625 \end{aligned}$$

- Remember in the Cournot competition game both firms were producing $q_i^* = 300$ in equilibrium, and both were obtaining a payoff of $90,000$.
- The reason why the first firm produces more is that
  - If firm 1 produces more firm 2 will produce less, and this is good for firm 1.
  - Firm 2 on the other hand is just best-responding to whatever firm 1 chooses to produce; as 1 produces more than 300 firm 2 will produce less than 300.
  - Firm 1 in Stackelberg game has first mover's advantage. It can predict firm 2's action and take advantage of it.

- Note also that the Cournot NE $(300, 300)$ is a NE outcome in the dynamic game: if firm 2 threats firm 1 by committing to produce 300 in the second stage, then firm 1 best response is to also produce 300
- However, this is not a credible threat: if firm 1 produces a different quantity, firm 2 will have to change her strategy according to best response function and thus the previous commitment was not realistic.
- Therefore, $(300, 300)$ is a NE but not a SPNE

Game Theory 106G
Dynamic Games with Imperfect Information

Menghan Xu

August 19, 2013

Dynamic Games with Unobserved Actions (Imperfect Information)

- If the actions of other players cannot be observed by their successor(s), then the successor may not be sure in what node they are
- For example, consider a Stackelberg competition where the quantity chosen by the first mover is unknown
- In this case the game will look like a normal Cournot game

Definition

The extensive form of a dynamic game with imperfect information consists of

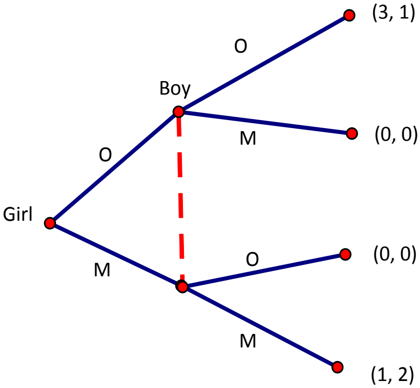
- A set of *players:* $I = \{1, 2, 3, \cdots, n\}$.
- The order of move represented by a *game tree - nodes and paths*
    - **Paths:** Each path represents an action.
    - **Action nodes:** Each action node specifies whose term to move. It extends to several actions that could be played by the player. It is reached by a single path (action taken by previous player);
    - **Terminal nodes:** Each terminal node is reached by a action taken by last mover. And it specifies payoffs of each player.
    - **Information Set:** Some action nodes are grouped together. In a node group, all nodes must belong to the same player and action paths extended from each node must be the same.

- For example, in the Stackelberg game with imperfect information, the information set for player 2 is composed by all the possible quantities that firm 1 may choose.
- Note that after firm 1 chooses a quantity, in firm 2's turn, his set of strategies is always the same
- Since a player in a information set has no idea which node he is standing on, his action is impossible to be contingent.

- Information sets are usually drawn in the trees with dashed lines connecting the nodes or a box containing the nodes
- Hence we can also use this generalization to model *simultaneous moves* in game trees.
- All that matters is that player i cannot make his action a_i contingent on the action a_j of another player j.

Example

Consider the Battle of the sexes, in terms of timing, the girl chooses O or M first. But the boy does not know the information, he chooses between O and M independently as well. So the game is equivalent to a simultaneous game.



- Here the information set is described by $\{O, M\}$ since the boy does not observe which action was taken by the girl
- The strategies are therefore $\{O, M\}$ for both players, i.e there are non-contingent
- We can no longer use backward induction here.

Subgames

- Due to the existence of information sets, we sometimes cannot use backward induction sometimes.
- We can still find SPNE of the game,
- Although we need to re-define subgame.

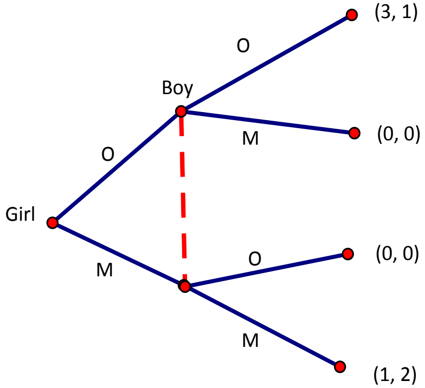
Definition

A subgame of a sequential game with imperfect information is a subset of the game, such that

- It has a single initial action node
- It contains all the nodes(action nodes and terminal nodes) and paths that are successors of the initial node.
- It contains all the nodes that are successors of any node it contains.
- If a node in a particular information set is in the subgame then all members of that information set belong to the subgame

Example

How many subgames in this game?



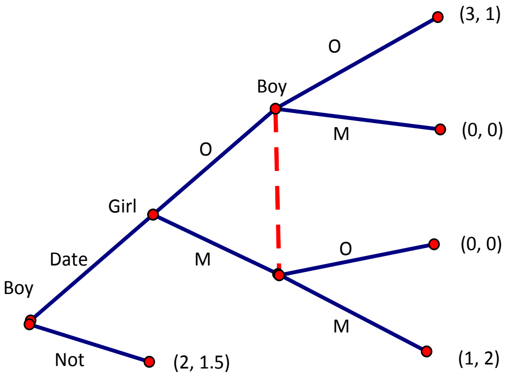
Example

Since each player has only 2 strategies, the sequential game is equivalent to

		Boy	
		O	M
Girl	O	3 , 1	0 , 0
	M	0 , 0	1 , 2

Example

Consider now a modified Battle of the Sexes where the boy can decide first wether he is going to date with the girl or not. If not, they will end up with $(2, 1.5)$ - their outside option...



- This game has only two subgames: the entire game and the game following the boy's decision to date, which is the original Battle of the Sexes game
- Whereas the strategies remain the same for the girl, the set of strategies for player 2 is now

$$\{(Date, O), (Date, M), (Not, O), (Not, M)\}$$

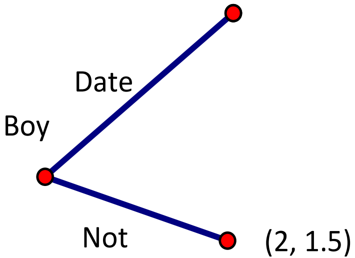
- We can transform the game into Normal form and find all pure Nash Equilibria

		Boy $(Date, O)$	$(Date, M)$	(Not, O)	(Not, M)
Girl	O	$(\underline{3}, 1)$	$(0, 0)$	$(\underline{2}, \underline{1.5})$	$(\underline{2}, \underline{1.5})$
	M	$(0, 0)$	$(\underline{1}, \underline{2})$	$(\underline{2}, 1.5)$	$(\underline{2}, 1.5)$

- *Can you find any mixed strategy NE by this matrix?

- To find all subgame perfect Nash Equilibria, we need to find Nash Equilibria of all subgames.
- The game can also be informally expressed as

		Boy	
		O	M
Girl	O	3 , 1	0 , 0
	M	0 , 0	1 , 2

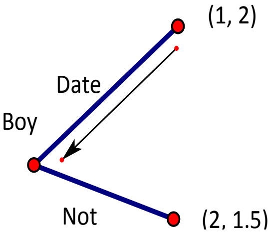


- The subgame

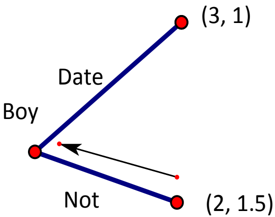
		Boy	
		O	M
Girl	O	3 , 1	0 , 0
	M	0 , 0	1 , 2

  is exactly the battle of sexes. So there are three Nash equilibria: (O, O) , (M, M), and a mixed strategy NE with $p = \frac{2}{3}$ and $q = \frac{1}{4}$
- We can choose any of them as the NE equilibrium for this subgame. Hence, there can be three subgame perfect Nash equilibria of the game. Two of them are pure SPNE.
- *and the third one is mixed strategy SPNE.

- If we focus on pure SPNE, there are two equilibria.
- If (M, M) is the NE result according to Boy's belief, he will chooses "Date" at first stage



- However, if (O, O) is the NE result according to Boy's belief, he will chooses "Not" at first stage



- Therefore, there are two (Pure) SPNE of the sequential game

$$\begin{array}{lll} 1.\text{Girl:} & O; \text{Boy:} & (Not, O) \\ 2.\text{Girl:} & M; \text{Boy:} & (Date, M) \end{array}$$

- The result is largely based on the boy's belief on which NE they will play at second stage.

Entry into an Monopolistic Market

- When we considered the entry game earlier we modeled the competition after entry by a single decision of the incumbent, to fight - yielding negative profits for both players - or cooperate - yielding positive profits for both players.
- Let us replace this very simple model of competition with Cournot competition.

Entry into an Monopolistic Market

- Players: Incumbent and Entrant
- Strategies:
  - $S_E = \{In, Out\} \times [0, \infty)$: The Entrant has to chooses whether or not to enter, and how much to produce if he entered
  - $S_I = [0, \infty) \times [0, \infty)$: The incumbent has to choose how much to produce if the entrant entered, q_I, and how much to produce if he did not, q_M.

- Payoffs: Assume that if a firm decides to produce they must pay a fixed cost equal to 10000. Per unit cost is still 100 (for both incumbent and entrant if enters) and inverse demand is given by $p(Q) = 1000 - Q$. Thus,
  - Payoff of the incumbent as a monopolist: $\pi_I = p(q_M)q_M - 100q_M - 10000$
  - Payoff of the incumbent in duopoly: $\pi_I = p(q_I + q_E)\, q_I - 100q_I - 10000$
  - Payoff of the entrant in duopoly: $\pi_E = p(q_I + q_E)\, q_E - 100q_E - 10000$
  - Payoff of the entrant when out: $\pi_E = 0$

- This game is not solvable with backward induction since both players move simultaneously in the Cournot “sub-game” while backward induction relied on the fact that the games considered could be solved step by step from end to the beginning.
- However, we can use the concept of SPNE by finding the NE of each subgame
- The game has three subgames, the Cournot competition subgame, the monopoly subgame, and the entire game

- In the Cournot competition game we know the solution is given by quantities $(300, 300)$ since the fixed costs do not affect response functions (remember the derivative of a constant is zero), and the equilibrium payoffs of the subgame is $(90000 - 10000, 90000 - 10000)$

$$(80000, 80000)$$

- In the Monopoly subgame when entrant decides to stay out, the incumbent chooses to produce 450 and gets a profit of $202500 - 10000 = 192500$, while the Entrant gets 0
- Given this, in the first stage the entrant would decide to enter the market since his payoff is greater

- Therefore the SPNE is given by the strategy profile $((In, 300), (300, 450))$
- Again note that the incumbent's strategy is contingent on the first choice of the Entrant (he specifies a number if Entrant chooses IN and another if Entrant chooses Out) but not on the Entrant's second choice (quantities) since the second stage is a simultaneous game
- Also note actions are optimal in every subgame in the sense that nobody would like to deviate, thus it is a NE

- We can also find other NE. But all of them are non-credible
- Consider the following strategy profile (s'_E, s'_I) where:
  - $s'_E = (Out, 50)$: The entrant does not enter (and would produce 50 if he had entered)
  - $s'_I = (800, 450)$: The incumbent floods the market by producing $q_I = 800$ if the entrant enters (in other words he Fights) and he produces his monopoly quantity of $q_M = 450$ if he is alone in the market.

- It is easy to check that this is an equilibrium
- The entrant receives a payoff of $\pi_E\left(s'_E, s'_I\right) = 0$ from this strategy profile
  - Staying *Out* and producing a different quantity q_E does not change his payoff
  - Changing his first action to *In* and choosing the optimal quantity $q_E = 50$ (given $q'_I = 800$) decreases his payoff to $\pi_E\left(s_E, s'_I\right) = (1000 - 800 - 50)\,50 - 100(50) - 10000 = 2500 - 10000 = -7500.$

- The incumbent receives a payoff of $\pi_I\left(s'_E, s'_I\right) = 450 * 450 - 10000 = 192500$ from this strategy profile.
  - Changing his first action (the quantity produced in oligopoly) does not change his payoff because this action node is not reached given the *Out* action of the entrant.
  - Changing his second action q_M can only decrease his payoff, because it is already chosen optimally.

- The second equilibrium is similar to the $(In, Fight)$ equilibrium in the initial entry game where the competition following entry was modeled as a single decision by the incumbent to "fight" or "cooperate".
- The reason why the incumbent chooses to flood the market by producing $q_I = 800$ is to threaten the entrant into staying out
- However,i it is not credible since the incumbent would never actually want to carry out the threat because it is unprofitable to himself.
- The idea of subgame-perfection is to rule out this knd of equilibrium that is relying on incredible threats, just like backward induction ruled out the incredible threat in the easier version of the game.

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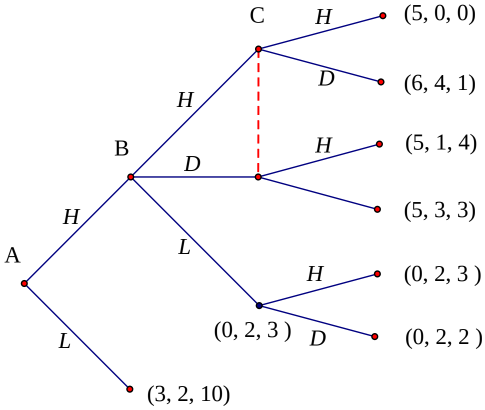
An comprehensive example

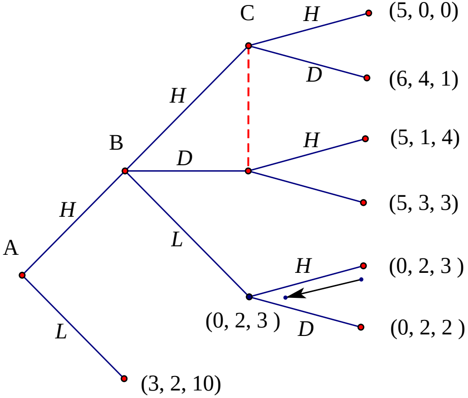
Example

Let's consider the following game with 3 players A, B, and C



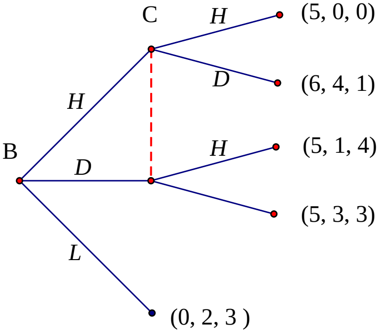
Example

The game have 3 subgames. By backward induction, we can highlight only one path. But nothing else can we do.



Example

Now consider the second subgame



- Notice that although it is a game with 3 players, this subgame is just for B and C. B has three strategies and C has two strategies.

Example

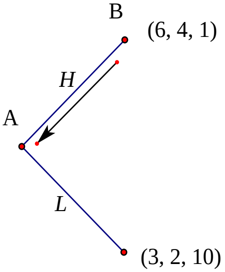
We can transform the game into normal form and find all NE.

B \ C	H	D
H	$(\ldots, 0, 0)$	$(\ldots, \underline{4}, \underline{1})$
D	$(\ldots, 1, \underline{4})$	$(\ldots, 3, 3)$
L	$(\ldots, \underline{2}, \underline{3})$	$(\ldots, 2, \underline{3})$

- There are two pure NE of the subgame. (Again, there can be mixed NE, but we ignore them)

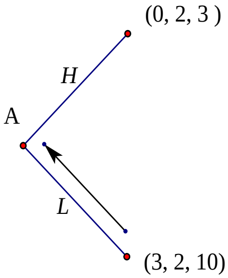
Example

If (H, D) is the NE played in the subgame, the game tree can be reduced to



Example

If (L, D) is the NE played in the subgame, the game tree can be reduced to



- We just find two SPNE of the game. For your own practice, state the equilibrium formally. (Don't forget off-equilibrium path)

Summer 2013
Game Theory 106G
Repeated Games

Menghan Xu

September 4, 2013

Repeated Games

- Repeated games with observed actions are dynamic games where:
  - In each stage, every player knows all the actions taken in previous stages
  - Players move simultaneously within each stage and play the same game repeatedly.
  - Such game within each stage is referred as the one-shot game or stage game
  - A pure strategy is a sequence of actions for each possible history of actions
  - Player's payoff is a function of the entire sequence of actions from the initial to the final stage, where the final stage can be taken to be infinite

- Repeated games have many subgames and even more strategies.
- Note that in repeated games two different histories lead to the same game in the last stage
- However, they still correspond to different subgames, and thus they may have a different NE

- Even for the simplest stage games, like the prisoners' dilemma, the number of strategies will grow exponentially fast.
- If such game is played twice, after first stage, there can be 4 differenct outcomes. Given each result, at second stage, each player has another two actions.
- Recall, strategy in dynamic game is a contingent plan which specify actions to do at each action node.

- Due to 4 different results in stage 1, there are four subgames.
- But among the four subgames in stage 2, only one will be reached by the actions in stage 1 eventually
- This subgame and its action nodes are said to be "on the equilibrium path".
- The other three subgames are not reached and are said to be "off the equilibrium path".
- We need to specify actions on off equilibrium path when we describe strategies.

- If a repeated game has finite stages, we can also use backward induction to find SPNE
- A SPNE implies that in the last stage players should play a NE of the stage game, even in the off equilibrium path last stage
- For the previous round we replace the last stage by one of its NE payoffs and then consider a NE of the simultaneous game where future payoffs have been added
- This process continues until we get to the initial node

An Example of Finitely Repeated Games: Prisoner's Dilemma

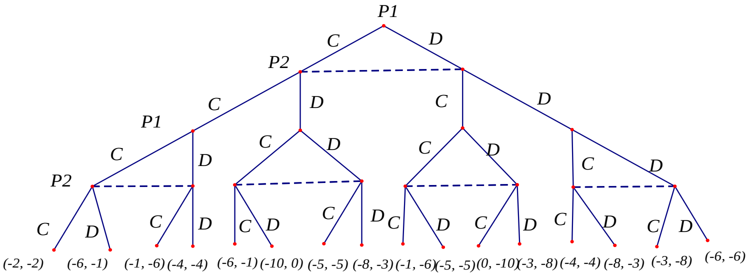
- Now let's consider a game in which prisoners' dilemma is played finite many times.
- The unique NE of prisoners' dilemma to play *Defect*, and the Pareto optimal situation is to play *Cooperate*. But in real life, it may not be always true.
- An explanation for playing *Cooperate* is that, a stage game is not the end of the story, and that they may meet again in the future in the same situation...

- Let's formalize this. Consider an extensive form game with two rounds:
  - Round 1: PD
  - Round 2: PD
- Therefore, since players can condition their actions on how their opponents play before, they could use later stages to punish their opponents

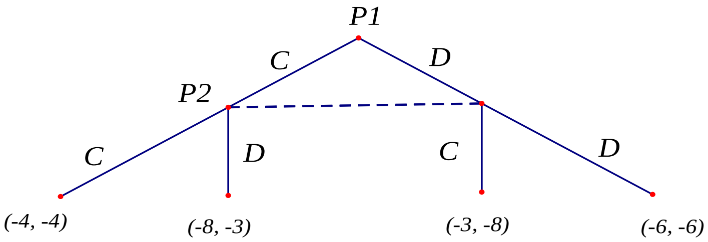
- The payoffs are additive across these two rounds of the game, each stage has the following payoffs

		2	
		C	D
1	C	$-1, -1$	$-5, 0$
	D	$0, -5$	$-3, -3$

- The game can be expressed by the tedious game tree



- There are totally 5 subgames of the game.
- To find SPNE, we start with the 4 subgames in second stage.
- As a prisoners' dilemma only has one Nash equilibrium (D, D) every player must choose D in each of the small subgames.



- The reduced game is equivalent to the 2×2 matrix

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{2} \\ & & C & D \\ \hline 1 & C & -4\,,\,-4 & -8\,,\,-3 \\ \hline & D & -3\,,\,-8 & -6\,,\,-6 \\ \hline \end{array}$$

- (D, D) is the unique Nash equilibrium of this game.
- Therefore the unique SPNE in the overall game is that each every always play D.

- *There are other Nash equilibria of the game.
- However,
  - none of them are subgame perfect
  - C is only played at off equilibrium paths
  - *Can you figure out one NE, in which C is played sometimes?

- We derive the SPNE by backward induction.
- In the last stage, Nash equilibrium strategies must be played.
- Since prisoner dilemma has unique Nash equilibrium, even if we repeated the PD 1000 times we would get the same outcome
- To get an SPNE in which C is played, we need to get rid of the **last round. - infinitely repeated game. (Next topic)**

Existence of SPNE

Theorem

In a repeated game, if $\{\sigma_i\}_{i=1,2...N}$ *is a Nash Equilibrium of the stage game, playing* $\{\sigma_i\}_{i=1,2...N}$ *at each period is a SPNE*

- If the stage game is a normal form game, then SPNE of the repeated game must exist

Theorem

If the stage game of a repeated game has more than 1 Nash Equilibrium, a strategy profile that "at each period, one of the Nash Equilibria is played" composes a subgame perfect Nash Equilibrium.

- For example, Battle of Sexes has 3 Nash Equilibria - 2 pure, 1 mixed. At each period, as long as one of them is played, the strategy profile is a SPNE.

- The above theorem guarantees the existence of SPNE in a repeated game.
- But the SPNE are composed by NE of its stage game.
- A more interesting question is that

  Is there any SPNE that non-NE strategies are played at some stage?

- The answer is positive.
- But notice that, at last period of the game, NE strategies must be played by backward induction.

Agreement and Punishment

- Consider the following game

		2	
		D	C
1	D	$-1\ ,\ -1$	$5\ ,\ 0$
	C	$0\ ,\ 5$	$4\ ,\ 4$

- Let's repeat it twice.

- Easy to see that, there are three Nash equilibria of the stage game
    - Pure: (D, C), (C, D)
    - Mixed: $\left(\frac{1}{2}C + \frac{1}{2}D, \frac{1}{2}C + \frac{1}{2}D\right)$, everybody plays each strategy with probability $\frac{1}{2}$
- Payoffs are $(5, 0)$, $(0, 5)$ and $(2, 2)$
- According to the theorem, any combination of this equilibrium strategies in the two periods constitutes a SPNE

Agreement with Punishment

- Always playing NE strategies is, to some extend, unconditonal on previous outcomes.
- To construct a SPNE in which non-NE actions are played, the two players have to make an "agreement"
- The agreement consists
  - What each player should do on equilibrium path
  - If anyone deviates from equilibrium path action, what they should do in on off equilibrium path
  - We may call the off equilibrium path action "punishment"

- Why such equilibrium is interesting?
  - Social Optimal
  - Pareto Improvement
  - (C, C) is not NE, but it generates the largest total payoff
- Potential problem of such agreement
  - No incentive to keep agreement - agreement is not profitable
  - No incentive to punish (Non-credible threat)

- Consider the following strategy profile
  - At first period, both players play C - first stage agreement
  - At second period, the agreement is a contingent plan conditional on what players have done at first stage.
    - If (C, C) was played, the players play the mixed strategy NE

$$\left(\frac{1}{2}D + \frac{1}{2}C, \frac{1}{2}D + \frac{1}{2}C\right)$$

    - If (D, C) was played, i.e. player 1 deviates, from agreement, (C, D) will be played. - Player 2 uses D to punish player 1.
    - If (C, D) was played, i.e. player 2 deviates from agreement, (D, C) will be played. - Player 1 uses D to punish player 2.
    - If (D, D) was played, the two players will play mixed strategy NE at second stage

- Why the above strategy is SPNE?
  - At second stage (last stage), in each subgame, a Nash equilibrium is played.
  - It guarantees that, player has incentive to punish.
  - Alternatively, you could say, an effective punishment must be a Nash Equilibrium
- At first stage, given the second period contingent agreement, we need to check whether C is a best response.
  - Alternatively, we need to check, whether either player has incentive to deviate from playing C to D

- If both players play C according to agreement
  - Each gets 4 at first period.
  - At second stage, they play mixed strategy NE, whose expected payoff is 2.
  - In total, the payoff on equilibrium path is $(4 + 2, 4 + 2) = (6, 6)$

- Assume player 1 deviates from C to D
  - At first period, player 1 earns 5.
  - At second stage, since player 2 will play D to punish him, player 1's best response is to player C.
  - In total player 1 will generate total payoff of $5 + 0 = 5$
  - Since $5 < 6$, player 1 has no incentive to deviate at first period
- The game is symmetric, player 2 has no incentive to deviate, too
- Therefore, the strategy profile is a subgame perfect Nash Equilibrium

- We never have to worry about the situation

Both players deviate from C to D

- Since Nash equilibrium is defined based on best responses.
- Players takes other players' strategies as given.

Ineffective Agreement

- The above example shows us a SPNE in which (C, C) is played, but sometimes, the agreement may not be effective
- Consider the following modified game

		2	
		C	D
1	C	-1 , -1	7 , 0
	D	0 , 7	4 , 4

- Let's repeat it twice.

- Easy to see that, there are three Nash equilibria of the stage game
  - Pure: (S, C), (C, S)
  - Mixed: $\left(\frac{3}{4}S + \frac{1}{2}C, \frac{1}{2}S + \frac{1}{2}C\right)$
- Payoffs are $(5, 0)$, $(0, 5)$ and $\left(\frac{3}{2}, \frac{3}{2}\right)$

- Let's construct an analogue of "agreement with punishment"
  - At first period, both players play C - first stage agreement
  - At second period, the agreement is a contingent plan conditional on what players have done at first stage.
    - If (C, C) is played, the players play the mixed strategy NE
    - If (D, C) is played, i.e. player 1 deviates, from agreement, (C, D) will be played. - Player 1 uses D to punish player 2.
    - If (C, D) is played, i.e. player 2 deviates from agreement, (D, C) will be played. - Player 2 uses D to punish player 1.
    - If (D, D) is played, the two players will play mixed strategy NE at second stage
- Does the agreement work this time?

- If both players play C according to agreement
  - Each gets 4 at first period.
  - At second stage, they play mixed strategy NE, whose expected payoff is $\frac{3}{2}$.
  - In total, the payoff on equilibrium path is $\left(4+\frac{3}{2}, 4+\frac{3}{2}\right)=\left(\frac{11}{2}, \frac{11}{2}\right)$

- Assume player 1 deviates from C to D
  - At first period, player 1 earns 7.
  - At second stage, since player 2 will play D to punish him, player 1's best response is to player C.
  - In total player 1 will generate total payoff of $7 + 0 = 7$
  - Since $5 > \frac{11}{2}$, player 1 does have incentive to deviate at first period
- Therefore, the strategy profile is no longer a SPNE
- Since deviating gives player higher total payoff than obeying the agreement.

Time value of money

- If a game has infinitely many periods, the total payoff will increase to infinity if we simply add them together.
- In reality, if you earn 1000 dollars per year, then the \$1000 today is different from \$1 ten years later
  - You cannot turn \$1000 in the future into consumption immediately
  - You can save the \$1000 today. Ten years later, it will becomes \$1000 plus a large amount of interest payment.
- Since money/payoff has "time value", we need to discount the future payoffs into its present value

Present value

- If you will be paid \$1000 one year later, but you need some money immediately.
- You borrow from your friend promising that you will pay him back \$1000 in one year
- If so, how much money the friend is willing to lend you?

- If the annual interest rate is $r > 0$, your friend is willing to lend you

$$\frac{1000}{1+r}$$

- Why?
- Since he can save the money to the bank, who will pay him

$$\frac{1000}{1+r} \times (1+r) = \$1000$$

one year later

- So the $\frac{1000}{1+r}$ is called the **present value** of 1000

- If you will be paid \$1000 two year later, how much money the friend is willing to lend you?
- Since r is annual interest rate, we need to discount the \$1000 twice
$$\frac{1000}{(1+r)^2}$$
- If the \$1000 will be paid N years later, the present value of it would be
$$\frac{1000}{(1+r)^N}$$
- If N goes to infinite, the present value becomes zero.

Discount Factor

- Instead of using interest rate, sometimes, we use discount factor to compute present value.
- Denote discount factor as $\delta \in [0, 1]$. Also,

$$\delta = \frac{1}{1 + r}$$

- If a \$1000 will be paid N periods later, the present value is

$$1000 \cdot \delta^N$$

- Notice that δ is always in $[0, 1]$.

Present value of a sequence of payment

Example

If you buy a financial product which pays you 5 today, 4 one year later and 6 three years later, what's the present value of the product? (Denote discount factor as δ)

$$PV = 5 + 4\delta + 6\delta^3$$

- Notice that we don't have to discount if the payoff is earned today.

Example

If you buy a financial product which pays you 5 each year forever. The first payment is paid today, what's the present value of the product?

$$PV = 5 + 5\delta + 5\delta^2 + 5\delta^3 \ldots$$

- It is a geometry sequence. You should know the following formular:

$$a + ar + ar^2 + \ldots ar^N = a\frac{1 - r^N}{1 - r}$$

- if $r < 1$ and N converges to infinity, we have

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots ar^N \ldots = \frac{a}{1 - r}$$

Example

Therefore, the present value of the financial product should be

$$PV = 5 + 5\delta + 5\delta^2 + 5\delta^3 ... = \frac{5}{1-\delta}$$

Example

If the first payment is paid one year later, the present value is

$$\begin{aligned} PV &= 5\delta + 5\delta^2 + 5\delta^3 ... \\ &= \frac{5\delta}{1-\delta} \end{aligned}$$

Example

In general If you are paid by $\$a$ for the first N periods and then a payoff of $\$b$ for the rest of the periods

$$
\begin{aligned}
PV &= a + a\delta + a\delta^2 + \ldots + a\delta^N + b\delta^{N+1} + b\delta^{N+2} + \ldots \\
&= a\left(\frac{1}{1-\delta} - \frac{\delta^{N+1}}{1-\delta}\right) + b\frac{\delta^{N+1}}{1-\delta} \\
&= a\frac{1-\delta^{N+1}}{1-\delta} + b\frac{\delta^{N+1}}{1-\delta}
\end{aligned}
$$

One-Shot Deviation

- Recall: Strategy is a contingent plan which specifies actions at all action nodes.
- When a repeated game has infinitely many periods, there so many possible istories and subgames - too many action nodes to discuss.
- If we want to check whether a strategy profile is SPNE, in principle there are infinitely many cases we need to check.
- But there is a theorem called "**One-shot Deviation Principle**" by which our workload of checking become dramatically smaller.

Definition

(One-shot deviation) For any repeated game (either finite or infinitely many periods), let $\{s_i(h^t)\}_{i=1,2,...N}$ is players' **strategy profile** that specifies their actions at time t conditional on the history of previous action history h^t. A one-shot deviation of player j at time T is an **action profile** (action path) $\{a_i^t\}_{i=1,2,...N}$ such that

$$a_j^T \neq s_j\left(h^T\right)$$

at T for player j. But for any other history h^t and any other player i,

$$a_i^t = s_i\left(h^t\right)$$

- Briefly speaking, one-shot deviation allows only ONE player deviates from orginal strategy "agreement" only ONCE!

Example

Consider the following game

		2	
		D	C
1	D	-1 , -1	5 , 0
	C	0 , 5	4 , 4

which is played for three times.

Example

consider the following strategy profile

- Start from playing (C, C)
- If (C, C) was always played at previous stages, players keep playing (C, C).
- Otherwise, players play (D, D).
- So the equilibrium path is

$$\{CCC, CCC\}$$

Example

Consider the following action paths, which are one-shot deviations?

- $$\{CCC, CCD\}$$

  It is a one-shot deviation: player 2 deviates once at third stage

- $$\{CCD, CCD\}$$

  It is not one-shot deviation, since both players deviate at third stage

Example

A few more examples

- $$\{CDD, CCD\}$$

  It is one-shot deviation: player 1 deviates at second stage. At third stage, according to agreement, (C, C) was not always played in previous stages, so they should play (D, D)

- $$\{CDC, CCD\}$$

  It is not one-shot deviation: player 1 deviates twice. At second stage, player 1 plays D instead of C. It is his first deviation. At stage 3, both players should play D, but A switches back to C - its his second deviation.

Example

Few more examples

- $$\{CDD, DDD\}$$

  It is a one-shot deviation. Player 2 deviates at first stage. At second stage, since $\{C, C\}$ was not played at stage 1, they should play (D, D). And third stage is the same.

- $$\{CDC, CCC\}$$

  It is not a one-shot deviation. Player 1 deviates at second stage. At third stage, they should play (D, D) but they turn back to (C, C)

- One-shot deviation means that there is only **ONE** player who deviates only **ONCE** from original strategy "agreement"
- A strategy "agreement" is contingent on history. Based on different history, actions could be different
- To check for one-shot deviation, we have to compare the action path with a strategy profile, instead of just its equilibrium path actions.

Example

Let's consider the same game, but the strategy is as follows

- Start from playing (C, D)
- If the action of last period is a pure Nash equilibrium actions of the stage game, they play (C, C)
- Otherwise, they play (D, C)

- So the equilibrium path is

$$\{CCD, DCC\}$$

Example

Let's consider the same game, but the strategy is as follows

- Start from playing (C, D)
- If the action of last period is a pure Nash equilibrium action of the stage game, they play (C, C)
- Otherwise, they play (D, C)

- So the equilibrium path is

$$\{CCD, DCC\}$$

Example

Consider the following actions,

- $$\{CDC, CCC\}$$

  It is a one-shot deviation. Player 2 deviates from D to C at first stage. So the first period is not a NE. At second stage, (D, C) should be played. Since (D, C) is a NE of the stage game, (C, C) should be played at stage three.

- $$\{CCD, DDC\}$$

  It is not a one-shot deviation. Since after first stage, (C, C) should be played, but player 2 deviates to play D. At third stage, since (C, D) was played at stage 2, (C, C) should be played. But player 1 deviates to D.

Profitable one-shot deviation

- As you've seen, there are so many ways to deviate from equilibrium. But we only care about one-shot deviations
- The reason why players want to deviate from original strategy profile is that the deviation may be profitable.

Example

Consider the game played 3 times

		2	
		D	C
1	D	-1 , -1	5 , 0
	C	0 , 5	4 , 4

Example

consider the following strategy profile

- Start from playing (C, C)
- If (C, C) was always played at previous stages, players keep playing (C, C).
- Otherwise, players play (D, D).
- So the equilibrium path is

$$\{CCC, CCC\}$$

- And the payoffs are

$$(4 + 4 + 4, 4 + 4 + 4) = (12, 12)$$

Example

Consider the one-shot deviations

- $$\{CCC, CCD\}$$

  In this example, the deviator is player 2. The payoffs are

  $$(4 + 4 + 0, 4 + 4 + 5) = (8, 13)$$

  Player 2 generates more payoff by the deviation. So we call it profitable one-shot deviation for player 2

Theorem

(One-shot deviation principle)
A strategy profile is SPNE if there is no profitable one-shot deviation from each player.

- To prove a strategy profile is SPNE, we need to just check all its one-shot deviations, in stead of all possible deviations

Example

Consider again,

		2	
		D	C
1	D	-1 , -1	5 , 0
	C	0 , 5	4 , 4

is played 3 times.

Example

The strategy profiles we are going to consider is as follows

- They play (C, C) at first stage.
- At second stage,
  - if (C, C) was played at t_1, they keep playing (C, C)
  - if (D, C) was played at t_1, (C, D) is played.
  - if (C, D) was played at t_1, (D, C) is played.
  - if (D, D) was played at t_1, (C, C) is played.
- At third stage,
  - If the history was $\{CC, CC\}$, they will play the mixed strategy NE
  - If the history was $\{CD, CC\}$, they play (C, D)
  - If the history was $\{CC, CD\}$, they play (D, C)
  - For all other histories, they play (C, C)

Example

The equilibrium path is

$$\{CCM, CCM\}$$

and payoffs are

$$(4+4+2, 4+4+2) = (10, 10)$$

- Since the game is symmetric, we assume the deviater is always player 1.

Example

- If he deviates at first stage, the action path becomes

$$\{DCC, CDD\}$$

the payoffs are

$$(5+0+0, 0+5+5) = (5, 10)$$

So deviating at first stage is not profitable

Example

- If he deviates at second stage, the action path becomes

$$\{CDC, CCD\}$$

the payoffs are

$$(4+5+0, 4+0+5) = (9, 9)$$

So deviating at second stage is not profitable, as well

- Since at last period, they play mixed strategy NE, deviation is not profitable for sure.
- In conclusion, since there is no profitable one-shot deviation of the strategy profile, it is a SPNE of the game.

Infinitely Repeated Games

- We've learnt two necessary tools in order to study infinitely repeated games
- Using discount factor to compute present values prevents the total payoff of the game becomes infinitely large
- One-shot deviation principle reduces number of cases we need to check for SPNE
- But some old tools are no longer useful - e.g. backward induction

Infinitely Repeated Prisoner Dilemma

		2: C	D
1	C	2 , 2	0 , 3
	D	3 , 0	1 , 1

- We already saw that if PD is finitely repeated, the unique SPNE is always choosing "Defect" since the stage has a unique NE
- If we repeat PD infinitely many times, always playing D is also a SPNE
- However this is not the unique one
- A more interesting question is: can we sustain C - cooperation in some SPNE now?

Grim-Trigger Strategy

- Yes, we can.
- But we still need an agreement with punishment to implement it.
- Consider the following strategy (called "grim-trigger strategy"):
    - At any time t, if no player ever played D in previous periods, both player play (C, C)
    - Otherwise, they play (D, D)
- It means that, on equilibrium path, both players keep choosing C,but as soon as one player defects once, (D, D) will be played forever.

- We need to use one-shot deviation principle to check whether the above strategy profile is SPNE.
- The key differences between infinitely repeated game and finite game is that, there is no end of the game. And you don't have to specify starting point action as well. Therefore, their could be two paths
    - Cooperation path

$$\{...CCCCC...,...CCCCC...\}$$

    - Defection path

$$\{...DDDDD...,...DDDDD...\}$$

- If the players are on the Defection path

$$\{...DDDDD..., ...DDDDD...\}$$

- The present value of the each player should be

$$1 + \delta + \delta^2 + \delta^3 ... = \frac{1}{1-\delta}$$

- If player 1 deviates to playing C at some stage, both players will play back to (D, D) afterwards,

$$\{...CDDDD..., ...DDDDD...\}$$

- And player 1's present value is

$$0 + \delta + \delta^2 + \delta^3 ... = \frac{\delta}{1-\delta}$$

which is strictly smaller than keeping playing D

- If the players are one the path where C is always played,

$$\{...CCCCC..., ...CCCCC...\}$$

- The present value of the each player should be

$$2 + 2\delta + 2\delta^2 + 2\delta^3... = \frac{2}{1-\delta}$$

- One shot deviation here means that player 1 deviates to D, and then both players play (D, D) for ever.

$$(...DDDDD..., CDDDD....)$$

  Then the expected payoff for player 1 is

$$3 + \delta + \delta^2 + \delta^3... = 3 + \frac{\delta}{1-\delta}$$

- Obviously, whether such deviation is profitable of not depends on what δ is.
- Non-profitable deviation means

$$\begin{aligned} \frac{2}{1-\delta} &\geq 3+\frac{\delta}{1-\delta} \\ \delta &\geq \frac{1}{2} \end{aligned}$$

- Notice that larger δ means future payoffs discount less. So players are more patient if δ is closer to one.
- If δ is small, players are impatient. Thus they will prefer an immediate 3, instead of a flow of 2.

Repeated Cournot Competition

- Counot competition, or quantity competition is very similar to prisoner dilemma.
- The cartel, in which firms work as a monopolist is not a stable situation.
- Now consider that the two firms repeatedly play Cournot game.
- Can we implement Grim-Trigger strategies to make a Cartel stable?
- The answer is yes, but when discount factor is large enough.

- Players: two firms A and B.
- Actions: $q_A \geq 0, q_B \geq 0$
- Market demand: $p = 100 - q_A - q_B$
- Unit cost: $c_A = c_B = 10$
- Discount factor $\delta \in (0, 1)$.

- Cournot competition outcome

$$(q_A, q_B) = (30, 30)$$

and payoffs are

$$(\pi_A, \pi_B) = (900, 900)$$

- Cartel outcome

$$\begin{aligned} Q &= 45 \\ q_A &= q_B = \frac{45}{2} \end{aligned}$$

and

$$\pi_A = \pi_B = 1012.5$$

- Cournot competition outcome

$$(q_A, q_B) = (30, 30)$$

and payoffs are

$$(\pi_A, \pi_B) = (900, 900)$$

- Cartel outcome

$$\begin{aligned} Q &= 45 \\ q_A &= q_B = \frac{45}{2} \end{aligned}$$

and

$$\pi_A = \pi_B = 1012.5$$

- Grim-Trigger Strategy
  - If the two firms keep colluding as a Cartel in the previous history, they keep the cartel
  - As long asa one player plays anything else, they start choosing Cournot competition quantities.
- Equilibrium payoff, for each player

$$1012.5 + 1012.5\delta + 1012.5\delta^2 + ... = \frac{1012.5}{1-\delta}$$

- One shot deviation
  - unlike PD, there are so many ways to deviate in quantity competition.
  - If player 1 decides to deviate, he knows that he has only one period of large benefit and will earn 900 forever afterwards.
  - So he will choose the most profitable deviation at the stage.
- The best response function of player 1 is

$$BR_1(q_2) = 45 - \frac{q_2}{2}$$

So the best deviation should be

$$q_1 = 45 - \frac{45}{4} = 33.75$$

- The profit at this period is

$$(100 - 33.75 - 22.5) \cdot 33.75 - 10 \cdot 33.75 = 1139.0625$$

- Therefore, the payoff of the deviation is

$$1139.0625 + 900\delta + 900\delta^2 + ... = 1139.0625 + \frac{900\delta}{1-\delta}$$

- To make the Grim-Trigger strategy a SPNE, the deviation should be non-profitable,

$$\begin{aligned} \frac{1012.5}{1-\delta} &\geq 1139.0625 + \frac{900\delta}{1-\delta} \\ 239.0625\delta &\geq 126.5625 \\ \delta &\geq 0.5294 \end{aligned}$$

- As long as $\delta \geq 0.5294$, even for the best deviation is non-profitable.

Summer 2013
Game Theory 106G
Information Economics

Menghan Xu

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Simultaneous Games with Asymmetric Info

- If somebody knows a piece of information that the others do not, we say the game has asymmetric information
- Such information may be
  - Player's type
  - Player's action
- Labor Market
  - Employees' qualities cannot be observed by employers
  - Employees' efforts cannot be observed by employers
- Buyer and seller
  - Buyers' taste is not observed by sellers
  - Sellers' good has unknown quality

- If one player cannot observe the other player's information, what will he do?
  - Maximize expected payoff
  - Design a mechanism to identify the other players' information

Example

(Market for Lemons)

- "**The Market for Lemons: Quality Uncertainty and the Market Mechanism**" is a 1970 paper by the economist George Akerlof. It discusses information asymmetry, which occurs when the seller knows more about a product than the buyer. A **lemon** is an American slang term for a car that is found to be defective only after it has been bought. (From wikipedia)

Example

(Market for Lemons)

- Players: seller s and buyer b
- Actions:

$$
\begin{aligned}
A_s &= \{\text{Sell, Not}\} \\
A_B &= \{\text{Buy, Not}\}
\end{aligned}
$$

Example

(Market for Lemons)

- There are two types of cars - Good or Bad(Lemons)
- Each type is equally likely ($q_g = q_b = 0.5$)
- A good car is worth $\$4,000$ to the seller and $\$5,000$ to the buyer.
- A bad car is worth $\$1,000$ to the seller and $\$2,000$ to the buyer.

Example

(Market for Lemons)

- If both players know the quality of the car, they will trade the two types of the car at p_g and p_b
- If the car is good, their payoffs are $(p_g - 4000, 5000 - p_g)$ if the car is bad, their payoffs are $(p_b - 1000, 2000 - p_b)$
- So the "reasonable" ranges for p_g and p_b are $[4000, 5000]$ and $[1000, 2000]$, respectively

- We call the situation "**first best**". Since there is no inefficiency caused by incomplete information.

Example

(Market for Lemons)

- If both players do not know the quality of the car, they will trade car at a single price $\hat{p}$
- They maximizes their expected payoffs
  - Buyer's expected payoff

$$(2000 - \hat{p}) \cdot q_b + (5000 - \hat{p}) \cdot q_g = 3500 - \hat{p}$$

  - Seller's expected payoff

$$(\hat{p} - 1000) \cdot q_b + (\hat{p} - 4000) \cdot q_g = \hat{p} - 2500$$

- So the "reasonable" ranges for $\hat{p}$ is $[2500, 3500]$

Example

(Market for Lemons)

- A more interesting question is that
  - What if the seller knows the quality of the car but the buyer does not?
- Types of seller:

$$T_s = \{\text{Good}, \text{Bad}\}$$

- Buyer's belief

$$\begin{aligned} \Pr(\text{Good}) &= 0.5 \\ \Pr(\text{Bad}) &= 0.5 \end{aligned}$$

Example

(Market for Lemons)

- Price: there can be only one price p
  - Assume there are two prices $p_b < p_g$, then seller will just claim that the car is good and sell at higher price
  - We assume the price p is a given number, not chosen by seller

Example

(Market for Lemons)

- Strategies
  - Seller: given price p and type of the car, he decides whether to sell or not
    - e.g.

$$\begin{aligned}&\text{Sell if good and sell if bad;}\\&\text{Sell if bad and not if good}\end{aligned}$$

  - Buyer: Given p and belief about the type, decide: Buy or not
- They make decisions simultaneously. The car is traded is and only if the seller decides to sell and the buyer decides to buy.
- When the car can be traded? We need to discuss different ranges of p.

Example

(Market for Lemons)

- If the car is actually good, seller is willing sell as long as $p \geq 4000$
- Buyer knows that
  - If $p \geq 4000$ and the seller is willing to sell, the car is a good car with probability $q_g = 0.5$.
  - Its expected payoff is
  $$0.5 \cdot 5000 + 0.5 \cdot 2000 = 3500$$
  So buyer will not accept it.
  - If $p < 4000$ and the seller is willing to sell, the car must be a lemon. So his payoff will be
  $$2000 - p$$
  and he will buy the can only when $p < 2000$

Example

- Thus, there is no price at which good cars are traded. If $p \in [1000, 2000]$, lemon cars will be traded.
- This phenomenon is known as **adverse selection**. The knowledge of the seller about the quality of the car has a negative influence on the (average) quality of the cars being traded. Good cars will never be chosen.
- Two issues make this happen:
  - There is too many lemons in the market. (Try $p_b = 0.2$)
  - The lemons are too bad (Try: Lemon is worth 3000 to seller and 4000 to buyer)

Definition

A (normal form) game with Incomplete Information is given by

- A set of players $i \in N$
- Types of players: $t_i \in T_i$, and a distribution of types $p_i\left(t_i\right)$ where $\sum_{i \in T_i} p_i\left(t_i\right) = 1$
- Action sets: A_i.
  Strategy sets: S_i. Each (pure) strategy $s_i : T_i \rightarrow A_i$ assigns some action $s_i\left(t_i\right) = a_i$ to every type t_i.
- Payoffs: $u_i(a_i, a_{-i}, t_i, t_{-i})$

Definition

A strategy profile $s^* = (s_1^*, s_2^*, ..., s_N^*)$ is a Bayesian Nash equilibrium (BNE) if each type t_i of each player i is playing a best response (in expectation) to the strategies of the others $s_{-i}^*(t_{-i})$

$$s_i^*(t_i) \text{ maximizes } \max_{a_i} E_{t_{-i}}\left[u_i(a_i, s_{-i}^*(t_{-i}), t_i, t_{-i})\right] \text{ for all } t_i$$

Note. This definition is nothing new. BNE is indeed just a Nash equilibrium if player i's strategy s_i is regarded as an action in the standard strategic game.

In the example of market for lemons

- Players: buyer and seller
- Type of players:
  - Only one type of buyer.
  - Two type of sellers - selling good car or selling lemons
- Action sets
  - Buyer: buy or not
  - Seller: sell or not
- Strategy set:
  - Buyer: buy or not
  - Seller: conditional on type, buy or sell

Example

(A Powerful Boss)

- Recall question 1b of homework 1 - if there is a boss who can save the prisoners.
- Both prisoners can be saved with probability p.
- If both are not released, their payoffs are according to typical PD game

		2: C	2: D
1	C	$(-1,-1)$	$(-5,0)$
	D	$(0,-5)$	$(-3,-3)$

- If one is released, his payoff is always zero.
- If one is not released but the other is released, his payoff is -1 if choosing C and -3 if choosing D

Example

(A Powerful Boss)

- Types: release or not
- If one is released, he needs to do nothing.
- So players' strategy is just $\{C, D\}$ conditional on type "not released", denoted as s_1^* and s_2^*
- If one is not released, he knows that the other prisoner is released with probability p and not with probability $1 - p$.

Example

(A Powerful Boss)

- From the perspective of player 1, if player 2's strategy is $s_2^* = D$, his expected payoff of choosing C is

$$-1 \cdot p + -5 \cdot (1 - p) = 4p - 5$$

  And expected payoff of choosing D is always -3.

- So player 1's best response is C if

$$\begin{aligned} 4p - 5 &\geq -3 \\ p &\geq \frac{1}{2} \end{aligned}$$

Example

(A Powerful Boss)

- However, from the perspective of player 2, if player 1's strategy is $s_1^* = C$, his expected payoff of choosing C is always -1, while expected payoff of choosing D is

$$-3p + 0 \cdot (1 - p) = -3p$$

- So player 2's best response is D if

$$\begin{aligned} -3p &\geq -1 \\ p &\leq \frac{1}{3} \end{aligned}$$

- No matter what p is, it is impossible for one player choosing D and the other choosing C in any equilibrium

Example

(A Powerful Boss)

- When $p < \frac{1}{3}$, both players will choose D.
  - When the other player is released with low probability, D is a better strategy since the game is more likely to be a prisoner dilemma.
- When $p \geq \frac{1}{2}$, both players will choose C.
  - When the other player is released with high probability, C is better.
- When $p \in \left[\frac{1}{3}, \frac{1}{2}\right]$, both (C, C) and (D, D) are Nash equilibrium solution.

Example

(Fighting unknown opponent)

- Suppose that two players can either engage in a fight over a resource or yield to the other player. If bother players fight, the stronger one wins.
- It is commonly known that player 1's strength is intermediate.
- Player 2's type is either strong or weak. The probability of being strong is p.
- Actions of both players are

$$\{F, Y\}$$

Example

(Fighting unknown opponent)

- Player 1's strategy set is

$$\{F, Y\}$$

- Player 2's strategy set is

$$\{FF, FY, YF, YY\}$$

first letter denotes his action if being strong; second letter denotes his action if being weak.

Example

(Fighting unknown opponent)

- If 2 is strong, the payoffs are summarized in the following matrix

$$\begin{array}{cc|c|c|} & & 2 & \\ & & F & Y \\ \hline 1 & F & (-1,1) & (1,0) \\ \hline & Y & (0,1) & (0,0) \\ \hline \end{array}$$

- If 2 is weak, we have

$$\begin{array}{cc|c|c|} & & 2 & \\ & & F & Y \\ \hline 1 & F & (1,-1) & (1,0) \\ \hline & Y & (0,1) & (0,0) \\ \hline \end{array}$$

Example

(Fighting unknown opponent)

- If player 2 is strong, F strictly dominates Y. So his strategy should be $F\_$.
- Given that player 2 chooses F is strong, the payoff of player 1 is
  - $(-1) \cdot p + (1-p) = 1 - 2p$ if chooses F
  - 0 if chooses Y

Example

(Fighting unknown opponent)

- If $p > 0.5$
  - Player 1 will choose Y.
  - Player 2 will choose F even if he is weak since player 1 yields anyway. So player 2's strategy is FF.
- If $p < 0.5$
  - Player 1 will choose F.
  - Player 2 will choose Y if he is weak. So player 2's strategy is FY.

Example

(Fighting unknown opponent)

- The result of the game depends on how likely the opponent is strong. If the probability is large, the player with no information advantage prefers to quit.
- The player with information advantage has a secured position. He never loses.

Example

(Cournot Competition with Unknown Costs)

- Consider the Cournot competition. But one firm's cost is unknown to the other firm.
- Formally,
  - Players: firm 1 and 2
  - Actions: q_1 and q_2
- Types:
  - Firm 1: Commonly known $c_1 = 10$
  - Firm 2: *H*igh cost type $c_2^H = 20$; *L*ow cost type $c_1^L = 0$
  - Each type can happen with equal probability
- Market demand: $p(Q) = 100 - Q$

Example

(Cournot Competition with Unknown Costs)

- Strategies
  - Firm 1: q_1
  - Firm 2: $\left(q_2^H, q_2^L\right)$
- Formally,
  - Players: firm 1 and 2
  - Actions: q_1 and q_2
- Types:
  - Player 1: Commonly known $c_1 = 10$
  - Player 2: *H*igh cost type $c_2^H = 20$; *L*ow cost type $c_1^L = 0$

Example

(Cournot Competition with Unknown Costs)

- Firm 1 chooses q_1 to maximize its expected payoff

$$\pi_1 = \frac{1}{2}\left(100 - q_1 - q_2^H\right) q_1 + \frac{1}{2}\left(100 - q_1 - q_2^L\right) q_1 - 10q_1$$

- Take derivative

$$\frac{\partial \pi_1}{\partial q_1} = 90 - 2q_1 - \frac{q_2^H + q_2^L}{2} = 0$$

- Best response function

$$q_1^* = 45 - \frac{q_2^H + q_2^L}{4}$$

Example

(Cournot Competition with Unknown Costs)

- Firm 2 chooses q_2^H to maximize its payoff

$$\pi_2^H = \left(100 - q_1 - q_2^H\right) q_2^H - 20 q_2^H$$

- Take derivative

$$\frac{\partial \pi_2^H}{\partial q_2^H} = 80 - 2q_2^H - q_1 = 0$$

- Best response function

$$q_2^{H*} = 40 - \frac{q_1}{2}$$

Example

(Cournot Competition with Unknown Costs)

- Similarly, the best response function of low type firm 2 is

$$q_2^{L*} = 50 - \frac{q_1}{2}$$

  (why?)

- Combine the 3 best response functions, we can easily solve for quantities

$$\begin{aligned} q_1^* &= 30 \\ q_2^{H*} &= 25 \\ q_2^{L*} &= 35 \end{aligned}$$

Summer 2013
Game Theory 106G
Auctions

Menghan Xu

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Incomplete information

- One of the great advantages of games with incomplete information is that we can model situations in which players do not know their opponents' preferences over outcomes.
- Some of the results we found for games with complete information hinged critically on the fact that each player exactly knew the preferences of the other players.
- For instance, in Bertrand competition we found that in Nash equilibrium a low-cost firm wants to price just at the marginal cost of the high cost firm.
- These results seem rather artificial, in particular when there is a large number of anonymous players, like bidders on eBay, and there is no way for any player to know the preferences of her opponents

Auctions

- There are N bidders, $i = 1, 2, 3....N$
- Each player has a value $v_i \in [0, 10]$ on the item on sale.
- The values are players' private information.
- Bidders believe that other bidders' values are identical and independent random variables following **uniform distribution**
  - The probability density function is a constant

$$f(v) = \frac{1}{10}$$

  - So

$$\Pr(v \le a) = \frac{a}{10}$$

- Strategy: each player conditional on his value, submit a bid $b_i(v_i)$ for the item.

Second Price Auction

- The bidder with highest bid wins the item.
- But pays the second highest price b^*
- Winner's payoff is

$$v_i - b^*$$

- Other bidders get zero.
- The known conclusion of Second Price Auction is that, no matter what the other bidders' values are, bidding true value is a weakly dominant strategy for each player.

- Recall the following table summarizes bidder $i's$ payoffs comparing bidding v_i and another $b_i \neq v_i$.
- If b_i is smaller than v_i

		Bidding v_i		Bidding b_i	
		Result	Utility	Result	Utility
1	$b_i < v_i < b^*$	Lose	0	Lose	0
2	$b_i < b^* < v_i$	Win	$v_i - b^* > 0$	Lose	0
3	$b^* < b_i < v_i$	Win	$v_i - b^* > 0$	Win	$v_i - b^* > 0$

- Although there may be more than one NE in this game, weak dominance at least helps us find one.

- If b_i is larger than v_i

		Bidding v_i		Bidding b_i	
		Result	Utility	Result	Utility
1	$v_i < b_i < b^*$	Lose	0	Lose	0
2	$v_i < b^* < b_i$	Lose	0	Win	$v_i - b^* < 0$
3	$b^* < v_i < b_i$	Win	$v_i - b^* > 0$	Win	$v_i - b^* > 0$

- We can see that, bidding v_i always weakly dominates other bids, no matter what other buyers bid.
- So $b_i^* (v_i) = v_i$ is a Nash Equilibrium strategy profile.

First Price Auction

- In first price auction, the bidder with highest bid wins the item.
- Different from second price auction, he pays his own bid b_i
- Winner's payoff is
$$u_i = v_i - b_i$$
- Since the winner has to pay his own bid, bidding one's own value is no longer a weakly dominant strategy. (Why?)
- Weakly dominant strategy does not even exist.

- Assume $N = 2$, two bidders
- Bidder 1's strategy is an optimal bidding function $b_1^* \left(v_1 \right)$ to maximize

$$E_{v_2} \left[u_i \left(b_1 ; b_2^* \left(v_2 \right), v_1, v_2 \right) \right] = \left(v_1 - b_1 \right) \Pr \left(b_1 \geq b_2^* \left(v_2 \right) \right)$$

- Assume bidder 1 believes that bidder 2 implements a **linear bidding rule**

$$b_2^* \left(v_2 \right) = a \cdot v_2$$

where $a \in \left(0, 1 \right)$.

- Bidder 1's expected payoff becomes

$$\begin{aligned}(v_1 - b_1)\Pr(b_1 \geq b_2^*(v_2)) &= (v_1 - b_1)\Pr(b_1 \geq a \cdot v_2) \\ &= (v_1 - b_1)\Pr\left(v_2 \leq \frac{b_1}{a}\right) \\ &= (v_1 - b_1) \cdot \frac{b_1}{10a}\end{aligned}$$

- So we can take derivative of above expression w.r.t b_1, let it equal to zero, and get

$$b_1^*(v_1) = \frac{1}{2}v_1$$

- The game is symmetric, we can claim that player 2 adopts the same bidding function - $1/2v_2$

- If there are N players, each player will bid

$$b_i^*\left(v_i\right) = \frac{N-1}{N} v_i$$

- *The above analysis is based on guess and verify, but actually we don't have to if you know the knowledge of differential equation.
- Such bidding function is the unique Bayes Nash Equilibrium of this game.

The two Auctions

- In second price auction
  - bidders bid their value
  - Bids are independent of distribution of value
  - Bids are independent of number of bidders
- In first price auction
  - Bidders bid below their value
  - Bids are conditional on value distribution
  - Bids conditional on number of bidders - more bidders, bid closer to own value

Revenue Equivalence

- From the perspective of the seller, he has to choose either 1st price auction or 2nd price auction.
- Which one will give him higher (expected) revenue?
- Assume there are two bidders,
- Under 2nd price auction, they will bid v_1 and v_2.
- If $v_1 > v_2$, the seller will earn v_2; if $v_1 < v_2$, the sell will earn v_1.
- The expected revenue is

$$\int_0^{10}\int_{v_2}^{10}\frac{v_2}{100}dv_1dv_2 + \int_0^{10}\int_0^{v_2}\frac{v_1}{100}dv_1dv_2$$

$$= \frac{10}{3}$$

- Under 2nd price auction, they will bid $v_1/2$ and $v_2/2$.
- If $v_1 > v_2$, the seller will earn $v_1/2$; if $v_1 < v_2$, the sell will earn $v_2/2$.
- The expected revenue is

$$\begin{aligned} & \int_0^{10}\int_{v_2}^{10}\frac{v_1}{200}dv_1dv_2+\int_0^{10}\int_0^{v_2}\frac{v_2}{200}dv_1dv_2 \\ = & \frac{10}{3} \end{aligned}$$

- You can see that, **the expected revenues are exactly the same**
- It is called "Revenue Equivalence"

Summer 2013
Game Theory 106G
Signaling

Menghan Xu

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Can we differentiate types?

- In the incomplete information examples, the player without information can only take one action. So he is in a big disadvantage.
- In many real cases, such players may have other powers.
- For instance,
    - A company cannot tell the type of workers. They can set requirements, e.g. degree
    - A insurance agent cannot tell the health condition of the buyer. They can set different packages and let the buyer choose by him self.
- Questions
    - Can we use some mechanism to distinguish types?
    - Is it profitable?

Job Market Signaling

- Players: a firm and a worker
- Worker: with different type H, L
- The probability of H and L are $1/3$ and $2/3$, respectively
- Workers can generate a college degree with different costs$\{c_H < c_L\}$ high type has lower education cost.

- The H type can generate 2000 to the firm
- The L type can generate 200 to the firm
- The market wage is regulated at 600
- Since
$$2000 \cdot \frac{1}{3} + 200 \cdot \frac{2}{3} - 600 = 200 > 0$$
It is profitable for the firm to hire a worker
- However, if the worker turns out to be L type, the firm suffers from loss.

- Now, assume that a firm can verify whether a worker has a college degree or not.
- For what $\{c_H, c_L\}$ can firm separate the two types of workers?
- It is easy to verify that as long as $c_H < 600$ and $c_L > 600$, the firm can use college degree to separate workers.
- And firm's expected payoff becomes

$$(2000 - 600) \cdot \frac{1}{3} = 1400/3 > 200$$

- Separating the workers is better.

Signaling with different jobs

- Players: a firm and a worker
- Worker: with different types H, L
- The probability of H and L are $1/3$ and $2/3$, respectively
- Workers can generate a college degree with costs

$$\begin{aligned} c_H &= 500 \\ c_L &= 900 \end{aligned}$$

high type has lower education cost.

- There are two types of jobs h and l.
- If H worker does h job, the revenue is 2000
- If L worker does h job, the revenue is 0
- Both workers generate 900 if they do l job
- To do h job, one must provide a college degree

- First, let's consider several packages of wages (w_h, w_l)
- $w_h = 1500, w_l = 500$
  - For H worker, his payoff of take h job is $1500 - 500 = 1000$, otherwise, he generate 500.
  - For L worker, his payoff of take h job is $1500 - 900 = 600$, otherwise, he generate 500.
- Easy to see that, the wages cannot separate workers and they will both choose h job
- Firm's expected payoff is

$$\frac{1}{3} \cdot 2000 - 1500 = -\frac{2500}{3}$$

- $w_h = 1100, w_l = 800$
  - For H worker, his payoff of take h job is $1100 - 500 = 600$, otherwise, he generate 800.
  - For L worker, his payoff of take h job is $1100 - 900 = 6200$, otherwise, he generate 800.
- Easy to see that, the wages cannot separate workers and they will both choose l job
- Firm's expected payoff is

$$900 - 800 = 100$$

- $w_h = 1500$, $w_l = 700$
  - For H worker, his payoff of take h job is $1500 - 500 = 1000$, otherwise, he generate 700.
  - For L worker, his payoff of take h job is $1500 - 900 = 600$, otherwise, he generate 700.
- In this case the wage is able to separate the workers - H worker chooses h job while L worker chooses l job.
- Firm's expected payoff is

$$\frac{1}{3}(2000 - 1500) + \frac{2}{3}(900 - 700) = 300$$

- There should be an optimal wage packages which maximizes firm's expected revenue.
- There are two questions we need to ask
  - What's the range of wages that can separate the two types of workers?
  - Is the optimal package to be separating? Or maybe not to separate is better.
- To answer the first question, we need two constraints

$$\begin{aligned} w_h - c_H &\geq w_l \\ w_l - c_L &< w_l \end{aligned}$$

- First constraint says that H type is willing to get a degree; the second constraint says that L type is not willing to get a degree.

- Combine the two constraints, we have

$$500 = c_H \leq w_h - w_l < c_L = 900$$

- So the optimal separating wage package is to set $w_l = 0$ and $w_h = 500$
- Turning to second question, is the package the best one?
- The package generates

$$\frac{1}{3}(2000 - 500) + \frac{2}{3} \cdot 900 = 1100$$

- Compare it with
  - $w_h - w_l > c_L = 900$ - $w_h = 900, w_l = 0$
    - Both workers get the degree. And firm's expected payoff is

$$\frac{1}{3}2000 - 900 = -\frac{700}{3}$$

  - $w_h - w_l < c_H = 500$ - $w_h = 0, w_l = 0$
    - Both workers do not get the degree. And the firm's expected payoff is 900
- Both are smaller than 1100. So the wage package $\{500, 0\}$ is indeed the optimal one.

- Is separating always good?
- Not necessarily
- It depends on
  - How costly is the education
  - How different are the jobs' benefits

- Players: a firm and a worker
- Worker: with different types H, L
- The probability of H and L are $1/3$ and $2/3$, respectively
- Workers can generate a college degree with costs

$$\begin{aligned} c_H &= 800 \\ c_L &= 900 \end{aligned}$$

high type has lower education cost.

- There are two types of jobs h and l.
- If H worker does h job, the revenue is 2000
- If L worker does h job, the revenue is 0
- Both workers generate 1300 if they do l job
- To do h job, one must provide a college degree

- To separate workers' type, we have

$$800 = c_H \leq w_h - w_l < c_L = 900$$

- So the optimal separating wage package is to set $w_l = 0$ and $w_h = 800$
- Expected payoff is

$$\frac{1}{3}(2000 - 800) + \frac{2}{3} \cdot 1300 = \frac{3800}{3}$$

- If firm just set $w_l = w_h = 0$, all workers will choose it and firm will generate 1300 for sure.
- Pooling is better